Mandl and Shaw reading assignments

Chapter 3 The Klein-Gordon Field

3.1 The Real Klein-Gordon Field ✔

3.2 The Complex Klein-Gordon Field

3.3 Covariant Commutation Relations

3.4 The Meson Propagator

Relativistic notations

$x^\mu = \{x^0, x^1, x^2, x^3\} = \{ct, x, y, z\}$

$x_\mu = \{x_0, x_1, x_2, x_3\} = \{ct, -x, -y, -z\} = g_{\mu\nu} x^\nu$

$g^{\mu\nu} = g_{\mu\nu} = \mathrm{DIAG}(1, -1, -1, -1)$

$A \cdot B = g_{\mu\nu} A^\mu B^\nu = g^{\mu\nu} A_\mu B_\nu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}$

(4D vec. product) (3D vec. product)

3.2 The Complex Klein-Gordon Field

■ Notations

$\hbar = 1$ and $c = 1$ (usually) See Section 6.1.

$\boldsymbol{x} = (t, \mathbf{x})$ or $x^\alpha = (x^0, x^1, x^2, x^3)$

$\varphi_{,\alpha} = \partial\varphi / \partial x^\alpha$ and $\varphi^{,\alpha} = \partial\varphi / \partial x_\alpha = g^{\alpha\beta} \varphi_{,\beta}$

■ Lagrangian field theory

$\pounds = 1/2\,(\varphi^*_{,\alpha} \varphi^{,\alpha} - \mu^2 \varphi^* \varphi)$ $\mu = mc/\hbar = m$

$(\Box + \mu^2)\,\varphi = 0$ and $(\Box + \mu^2)\,\varphi^* = 0$

$\pi(\boldsymbol{x}) = \partial\varphi^* / \partial t$ and $\pi^*(\boldsymbol{x}) = \partial\varphi / \partial t$

■ Equal-time commutation relations

$[\,\varphi(t,\mathbf{x}), \varphi(t,\mathbf{x}')\,] = 0$

$[\,\varphi(t,\mathbf{x}), \pi(t,\mathbf{x}')\,] = i\hbar c^2\, \delta^3(\mathbf{x} - \mathbf{x}')$

etc.

Fourier expansion

$$\varphi(\boldsymbol{x}) = \sum_{\mathbf{k}} N \left(a_{\mathbf{k}}\, e^{-i\boldsymbol{k}.\boldsymbol{x}} + b^{\dagger}_{\mathbf{k}}\, e^{+i\boldsymbol{k}.\boldsymbol{x}} \right)$$

where $N = (1 / 2\Omega\omega)^{1/2}$

and $\boldsymbol{k}.\boldsymbol{x} = k_\mu x^\mu = \omega t - \mathbf{k}.\mathbf{x} = k^0 x^0 - k^i x^i$
(*very flexible but involved notations !*)

Creation and annihilation operators

$[a_{\mathbf{k}}, a_{\mathbf{k'}}] = 0$ *and* $[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k'}}] = \delta_{\mathbf{k},\mathbf{k'}}$

$[b_{\mathbf{k}}, b_{\mathbf{k'}}] = 0$ *and* $[b_{\mathbf{k}}, b^{\dagger}_{\mathbf{k'}}] = \delta_{\mathbf{k},\mathbf{k'}}$

$[a_{\mathbf{k}}, b_{\mathbf{k'}}] = 0$ *and* $[a_{\mathbf{k}}, b^{\dagger}_{\mathbf{k'}}] = 0$

i.e., there are two different mesons –

particle ($a^{\dagger}_{\mathbf{k'}}$) and antiparticle ($b^{\dagger}_{\mathbf{k'}}$)

Important operators

$P^\alpha = (H/c,\mathbf{P})$ = the 4-momentum operator

P^α = *See 3.12 and 3.13;* $H = \int \boldsymbol{H}(\pi,\varphi)\, d^3x$

$P^\alpha = \sum_k \hbar k^\alpha (N_a(\mathbf{k}) + N_b(\mathbf{k}))$

Q = the charge operator

Q = *See 3.31* ; ***NEEDS ELECTROMAGNETISM TO INTERPRET***

$Q = q \sum_k (N_a(\mathbf{k}) - N_b(\mathbf{k}))$ ***opposite charges***

Real (or, hermitian) field operators

We could define $\varphi_1(x)$ and $\varphi_2(x)$ by

$\varphi_1 = 1/\sqrt{2}\, (\varphi + \varphi^{\dagger})$ and $\varphi_2 = -i/\sqrt{2}\, (\varphi - \varphi^{\dagger})$;

these imply

$\varphi = 1/\sqrt{2}\, (\varphi_1 + i\, \varphi_2)$ and $\varphi^{\dagger} = 1/\sqrt{2}\, (\varphi_1 - i\, \varphi_2)$.

$\therefore$ ***The complex scalar field is equivalent to a theory of two real fields***

3.3 Covariant Commutation Relations

Mandl and Shaw: For a ***real*** scalar field, define

$$[\,\varphi(\boldsymbol{x})\,,\,\varphi(\boldsymbol{y})\,] = i\,\hbar c\,\Delta(\,\boldsymbol{x}-\boldsymbol{y}\,);$$

($\boldsymbol{x}$ means x^μ)

then

$$\Delta(\xi) = -\,c\int d^3k/(2\pi)^3\ \frac{\sin(\boldsymbol{k}.\xi)}{\omega_{\mathbf{k}}}\ .$$

That is,

$$\Delta(\xi^\mu) = -\,c\int d^3k/(2\pi)^3\ \frac{\sin(\omega\xi^0-\mathbf{k}.\xi)}{\omega_{\mathbf{k}}}$$

where $\omega = c\sqrt{(|\mathbf{k}|^2+\mu^2)}$.

Here let's consider the ***complex*** scalar field. Define

$$[\,\varphi(\boldsymbol{x})\,,\,\varphi^\dagger(\boldsymbol{y})\,] = i\,\hbar c\,\Delta(\,\boldsymbol{x}-\boldsymbol{y}\,).$$

CALCULATION (Set $\hbar$ = 1 and c = 1 .)

$$[\,\varphi(\boldsymbol{x})\,,\,\varphi^\dagger(\boldsymbol{y})\,] = \sum N \sum' N'$$
$$[\ (a_{\mathbf{k}}\, e^{-ik.x} + b^\dagger_{\mathbf{k}}\, e^{+ik.x}),$$
$$(a^\dagger_{\mathbf{k}'}\, e^{+ik'.y} + b_{\mathbf{k}'}\, e^{-ik'.y})\ ]$$

$$= \sum N \sum' N'\ \{\,\delta(\mathbf{k},\mathbf{k}')\, e^{-ik.(x-y)}$$
$$-\,\delta(\mathbf{k},\mathbf{k}')\, e^{+ik.(x-y)}\ \}$$

$$= \sum N^2\,(-2i)\,\sin(\,k.(x-y)\,)$$

$$= \int d^3k/(2\pi)^3\ \ 1/(2\omega)\ \ (-2i)\,\sin(\,k.(x-y)\,)$$

$$= i\,\Delta(\,\boldsymbol{x}-\boldsymbol{y}\,)$$

where again

$$\Delta(\xi) = -\int d^3k/(2\pi)^3\ \frac{\sin(k.\xi)}{\omega_{\mathbf{k}}}\ .$$

$k.\xi = \omega\xi^0 - \mathbf{k}.\xi$

COVARIANCE OF THE COMMUTATION RELATION *($\hbar$=1 and c=1)*

THEOREM.

$$\Delta(\xi) = -\, i/(2\pi)^3 \int d4k\ \delta(k.k - \mu^2)\ \varepsilon(k^0)\ e^{-ik.\xi}$$

where $\varepsilon(k^0) = \pm 1$ for $k^0 \gtrless 0$. *[sign fun.]*

PROOF.

$$\delta(k^2 - \mu^2) = \delta[\,(k^0)^2 - \omega^2\,]$$
$$= \delta[\,(k^0 - \omega)(k^0 + \omega)\,]$$
$$= \delta[\,k^0 - \omega\,]/2\omega + \delta[\,k^0 + \omega\,]/2\omega$$
$$= 1/(2\omega)\,\{\,\delta[\,k^0 - \omega\,] + \delta[\,k^0 + \omega\,]\,\}$$

So,

$$\Delta(\xi) = -\, i/(2\pi)^3 \int d^3k\ 1/(2\omega)\ e^{-i(\omega t - \mathbf{k}.\xi)}$$
$$-\, i/(2\pi)^3 \int d^3k\ (-1)/(2\omega)\ e^{-i(-\omega t - \mathbf{k}.\xi)}$$

$$= -\, i/(2\pi)^3 \int d^3k\ /(2\omega)\ e^{-i(\omega t - \mathbf{k}.\xi)}$$
$$+\, i/(2\pi)^3 \int d^3k\ /(2\omega)\ e^{-i(-\omega t + \mathbf{k}.\xi)}$$
$$= -\, i/(2\pi)^3 \int d^3k\ /(2\omega)\ e^{-i(\omega t - \mathbf{k}.\xi)}$$
$$+\, i/(2\pi)^3 \int d^3k\ /(2\omega)\ e^{+i(\omega t - \mathbf{k}.\xi)}$$
$$= +\, i \int d^3k/(2\pi)^3\ /(2\omega)\ 2i \sin(\omega t - \mathbf{k}.\xi)$$

$$= -\int d^3k/(2\pi)^3\ \frac{\sin(k.\xi)}{\omega_{\mathbf{k}}}\ . \qquad \text{Q.E.D.}$$

The theorem shows that [$\varphi(x)$, $\varphi(y)$] is covariant : because the integral on the right-hand side is *manifestly* covariant.

Microcausality

[$\varphi(x^\mu)$, $\varphi(y^\mu)$] = 0 if $(x - y)^2 < 0$,

i.e., if x and y have spacelike separation.

Proof: covariance implies we can set $x^0 = y^0$; and the ETCR is 0.

Calculate the commutator function

$$\phi(x^\mu) = \sum_{\vec{k}} N \left(a_{\vec{k}}\, e^{-ik\cdot x} + a^{+}_{\vec{k}}\, e^{ik\cdot x} \right)$$

$$[\phi(x_1), \phi(x_2)] = \sum_{k_1}\sum_{k_2} N_1 N_2 \Big\{ e^{-ik_1 x_1} e^{ik_2 x_2} [a_{k_1}, a^{+}_{k_2}] + e^{ik_1 x_1} e^{-ik_2 x_2} [a^{+}_{k_1}, a_{k_2}] \Big\}$$

$$= \sum_{k_1} N_1^2 \left(e^{-ik_1\cdot(x_1-x_2)} - e^{ik_1\cdot(x_1-x_2)} \right)$$

Let $\xi^\mu = x_1^\mu - x_2^\mu$

$$N_1^2 = \frac{1}{2\omega\Omega}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \left[e^{-ik\cdot\xi} - e^{ik\cdot\xi} \right]$$

- For spacelike separation, let $\xi^0 = 0$;

$$[\phi(x_1), \phi(x_2)] = \int \frac{d^3k}{(2\pi)^3 \cdot 2\omega} \Big[\underbrace{e^{i\vec{k}\cdot\vec{\xi}} - e^{-i\vec{k}\cdot\vec{\xi}}}_{\text{odd function of } \vec{k}} \Big]$$

$$= 0$$

- For timelike separation, let $\vec{\xi} = 0$;

$$[\phi(x_1), \phi(x_2)] = \int \frac{d^3k}{(2\pi)^3 \cdot 2\omega} \left(e^{-i\omega\xi^0} - e^{i\omega\xi^0} \right)$$

Let $\xi^0 = \tau/m$; i.e., $\tau = m\xi^0$ (dimensionless)

$$[\phi(x_1), \phi(x_2)] = g(\tau) - g(-\tau)$$

where $g(\tau) = \int \frac{d^3k}{(2\pi)^3 \cdot 2\omega} e^{-i\frac{\omega}{m}\tau}$

$$= \int \frac{4\pi k^2 dk}{(2\pi)^3 \cdot 2\omega} e^{-i\frac{\omega}{m}\tau} \quad \text{where} \quad \omega = \sqrt{k^2+m^2}$$

$$\omega\, d\omega = k\, dk$$

$$\frac{k^2 dk}{\omega} = k\, d\omega$$

$$= \frac{1}{(2\pi)^2} \int_m^\infty \sqrt{\omega^2 - m^2}\, d\omega\, e^{-i\frac{\omega}{m}\tau}$$

Let $\omega = m u$

$$= \frac{m^2}{(2\pi)^2} \int_1^\infty \sqrt{u^2-1}\, du\, e^{-iu\tau} = \frac{m^2}{(2\pi)^2} G(\tau)$$

Now we use some Bessel function magic

$$G(\tau) = \int_1^\infty \sqrt{u^2-1}\, du\, e^{-iu\tau} \qquad \frac{u^2-1}{\sqrt{u^2-1}}$$

$$= \left(-\frac{d^2}{d\tau^2} - 1\right) \underbrace{\int_1^\infty \frac{du}{\sqrt{u^2-1}}\, e^{-iu\tau}}_{H_0^{(2)}(\tau)\cdot\frac{\pi}{2i}}$$

$$= \frac{i\pi}{2}\left(\frac{d^2 H_0^{(2)}}{d\tau^2} + H_0^{(2)}\right)$$

$$= \frac{i\pi}{2}\left(\frac{-1}{\tau}\frac{dH_0^{(2)}}{d\tau}\right) \quad \text{by Bessel's equation}$$

$$= \frac{i\pi}{2\tau} H_1^{(1)}(\tau) \quad \text{(Hankel function identity)}$$

$$G(\tau) - G(-\tau) = \frac{i\pi}{2\tau}\left[H_1^{(1)}(\tau) - H_1^{(1)}(-\tau)\right]$$

$$= \frac{i\pi}{2\tau}\left[H_1^{(1)}(\tau) - H_1^{(2)}(\tau)\right] = \frac{i\pi}{2\tau}\, 2i\, Y_1(\tau)$$

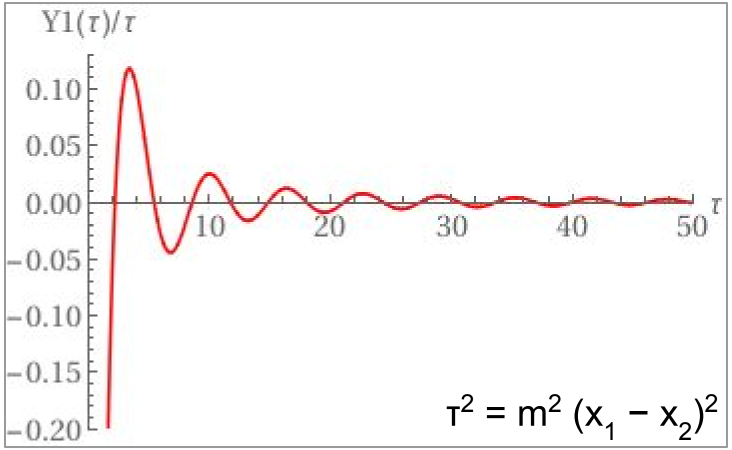
$$= -\frac{\pi}{\tau} Y_1(\tau) \qquad \text{BesselY}$$

For a timelike separation,

$$[\phi(x_1), \phi(x_2)] = \frac{m^2}{(2\pi)^2}\left(-\frac{\pi}{\tau} Y_1(\tau)\right)$$

$$\tau = m\xi^0 = m\sqrt{\xi^2}$$

$$= \frac{-m}{4\pi\sqrt{(x_1-x_2)^2}}\, Y_1\left(m\sqrt{(x_1-x_2)^2}\right)$$



3.4 THE MESON PROPAGATOR

Mandl and Shaw: for a ***real*** scalar field, define

$<0|\ T\{\ \varphi(\boldsymbol{x})\ \varphi(\boldsymbol{y})\ \}|0> \ = i\,\hbar c\ \Delta_F(\boldsymbol{x}-\boldsymbol{y});$

then

$$\Delta_F(\xi) = \int_{CF} d^4k/(2\pi)^4 \ \frac{e^{-ik.x}}{(k^2-\mu^2)} \ ,$$

where the contour *CF* is appropriately chosen.

Let's consider instead the complex field. Define the propagator by

$<0|\ T\{\ \varphi(\boldsymbol{x})\ \varphi^{*}(\boldsymbol{y})\ \}\ |0> = i\,\hbar c\ \Delta_F(\boldsymbol{x}-\boldsymbol{y}).$

Now derive the function $\Delta_F(\boldsymbol{x}-\boldsymbol{y})$.

There are several ways to derive the result.

Define

$$i\,\hbar c\ \Delta^{+}(x-y) = <0|\ \varphi(x)\ \varphi^{*}(y)\ |0>$$

$$i\,\hbar c\ \Delta^{-}(x-y) = -<0|\ \varphi^{*}(y)\ \varphi(x)\ |0>$$

Note that

$$i\,\hbar c\ (\ \Delta^{+}(x-y) + \Delta^{-}(x-y)\ )$$

$$= <0|\ [\ \varphi(x)\ ,\ \varphi^{*}(y)\ ]\ |0>$$

$$= <0|\ i\,\hbar c\ \Delta(x-y)\ |0>$$

$$= i\,\hbar c\ \Delta(x-y)$$

$$\boxed{\Delta^{+}(x-y) + \Delta^{-}(x-y) = \Delta(x-y)}$$

Now the propagator ...

$$<0|\ T\{\ \varphi(x)\ \varphi^{\dagger}(y)\ \}\ |0> = i\,\hbar c\ \Delta_F(x-y)$$

$$= <0|\ \theta(x^0-y^0)\ \varphi(x)\ \varphi^{\dagger}(y)$$

$$+\ \theta(y^0-x^0)\ \varphi^{\dagger}(y)\ \varphi(x)\ |0>$$

$$= \theta(x^0-y^0)\ i\,\hbar c\ \Delta^{+}(x-y)$$

$$+\ \theta(y^0-x^0)\ (-i\,\hbar c)\ \Delta^{-}(x-y)$$

$$\boxed{\begin{aligned}\Delta_F(\,x-y\,) &= \theta(x^0-y^0)\ \Delta^{+}(\,x-y\,)\\ &\quad -\theta(y^0-x^0)\ \Delta^{-}(\,x-y\,)\end{aligned}}$$

FOUR FUNCTIONS *(check these eqs. in Section 3.4)*

$$\Delta(\xi) = -\int d^3k/(2\pi)^3\ \frac{\sin(k.\xi)}{\omega_{\mathbf{k}}}$$

$$\Delta^{+}(\xi) = \int d^3k/(2\pi)^3\ \frac{i\exp(\,i\,k.\xi)}{2\,\omega_{\mathbf{k}}}$$

$$\Delta^{-}(\xi) = \int d^3k/(2\pi)^3\ \frac{(-i)\exp(\,-ik.\xi)}{2\,\omega_{\mathbf{k}}}$$

$$\Delta_F(\xi) = \int d^3k/(2\pi)^3\ \frac{i}{2\,\omega_{\mathbf{k}}}$$

$$\times\ \{\ \theta(\xi^0)\exp(\,i\,k.\xi) + \theta(-\xi^0)\exp(\,-i\,k.\xi)\ \}$$

THEOREM

$$\Delta_F(\xi) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-i\,k.\xi}}{k^2 - \mu^2 + i\,\varepsilon}$$

This is ***the crucial equation*** for understanding Feynman diagrams.

Prove the theorem next time.

Homework Problems due Friday Feb 10

Problem 16.

Equal time commutation relations.

We have, in the Schroedinger picture,

$$[\,\psi(\mathbf{x})\,,\,\psi^\dagger(\mathbf{x}')\,] = \delta^3(\mathbf{x}-\mathbf{x}')\quad,$$

$$[\,\psi(\mathbf{x})\,,\,\psi(\mathbf{x}')\,] = 0\ \text{, etc.}$$

(a) Show that in the Heisenberg picture, this commutation relation holds at all equal times.

(b) What is the commutation relation for *different* times?

Problem 17.

(a) Do problem 2.1 in Mandl and Shaw.

(b) Do problem 2.2 in Mandl and Shaw.

(c) Do problem 2.3 in Mandl and Shaw.

Problem 18.

For the free real scalar field,

(A) Write H in terms of $\pi(\mathbf{x})$ and $\varphi(\mathbf{x})$.

(B) Write H in terms of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$.

Problem 19.

(A) Mandl and Shaw problem 3.3.

(B) Mandl and Shaw problem 3.4.

Problem 20.

The Yukawa theory problem.