Mandl and Shaw reading assignments

Chapter 2 Lagrangian Field Theory

2.1 Relativistic notation ✔

2.2 Classical Lagrangian field theory ✔

2.3 Quantized Lagrangian field theory ✔

2.4 Symmetries and conservation laws ✔

Problems; 2.1 2.2 2.3 2.4 2.5

Chapter 3 The Klein-Gordon Field

3.1 The real Klein-Gordon field ✔

3.2 The complex Klein-Gordon field ✔

3.3 Covariant commutation relations

3.4 The meson propagator

Problems; 3.1 3.2 3.3 3.4 3.5

WE NOW KNOW THESE THESE TOPICS

- Lagrangian field theory
  $\mathcal{L} \rightarrow$ field equation(s)

- Real and complex scalar fields
  Klein-Gordon equation

- Covariant commutator
  $[\, \varphi(x_1)\, ,\, \varphi(x_2)\,]$

- Propagator *(not quite !)*
  $\langle 0 |\; T \;\; \varphi(x_1) \;\; \varphi(x_2) \; | 0 \rangle$

MANDL and SHAW, Chapters 2 , 3

MAIANI and BENHAR, Chapters 3 , 4

M&Sh - Chapter 3 - The Klein-Gordon Field

3.1 The Real Klein-Gordon Field ✔

3.2 The Complex Klein-Gordon Field ✔

3.3 Covariant Commutation Relations ✔

3.4 The Meson Propagator

Heaviside Theta Function

$$\theta(s) = \begin{cases} 1 & \text{if} \quad s > 0 \\ 0 & \text{if} \quad s < 0 \end{cases}$$

also called the step function.

3.4 THE FEYNMAN PROPAGATOR

for a complex (ie., non-hermitian) scalar field.

■ DEFINITION OF THE PROPAGATOR

$$i\hbar c\, \Delta_F(\boldsymbol{x} - \boldsymbol{y}) = \langle 0 |\, T\, \varphi(\boldsymbol{x})\, \varphi^\dagger(\boldsymbol{y})\, | 0 \rangle$$

Set ħ = 1 and c = 1, so

$$i\, \Delta_F(\boldsymbol{x} - \boldsymbol{y}) = \theta(x^0 - y^0) \langle 0 |\, \varphi(\boldsymbol{x})\, \varphi^\dagger(\boldsymbol{y})\, | 0 \rangle$$
$$+ \;\; \theta(y^0 - x^0) \langle 0 |\, \varphi^\dagger(\boldsymbol{y})\, \varphi(\boldsymbol{x})\, | 0 \rangle$$

Recall, for the *free field*,

$$\varphi(x^\mu) = \sum_{\mathbf{k}} N \left(e^{-ik.x}\, a_{\mathbf{k}} + e^{ik.x}\, b_{\mathbf{k}}^\dagger \right)$$

$$N = \sqrt{[\, \hbar/(2\omega V)\,]} = \sqrt{[\, 1/(2\omega V)\,]}$$

$$k.x = \omega t - \mathbf{k.x} = \omega t - k^i x^i \qquad (\, i = 1\ 2\ 3\,)$$

$$\omega = \omega_{\mathbf{k}} = \sqrt{(\, k^i k^i + \mu^2\,)} \quad ; \quad \mu = mc/\hbar = m$$

■ CALCULATION OF THE PROPAGATOR

$$i\,\Delta_F(\boldsymbol{x}-\boldsymbol{y}) = \theta(x^0 - y^0)\ \langle 0|\ \varphi(\boldsymbol{x})\,\varphi^{\dagger}(\boldsymbol{y})\ |0\rangle$$
$$+\ \theta(y^0 - x^0)\ \langle 0|\,\varphi^{\dagger}(\boldsymbol{y})\,\varphi(\boldsymbol{x})\ |0\rangle$$

Start with $\mathbf{A_1} = \langle 0|\ \varphi(\boldsymbol{x})\,\varphi^{\dagger}(\boldsymbol{y})\ |0\rangle$

$$= \sum N \sum' N'$$
$$\langle 0|\ (e^{-ik.x}\, a_{\mathbf{k}} + e^{ik.x}\, b_{\mathbf{k}}^{\dagger})$$
$$(e^{ik'.y}\, a_{\mathbf{k'}}^{\dagger} + e^{-ik'.y}\, b_{\mathbf{k}})\ |0\rangle$$

Note

$$b_{\mathbf{k}}|0\rangle = 0 \quad \text{and} \quad \langle 0|b_{\mathbf{k}}^{\dagger},$$

so

$$A_1 = \sum N \sum' N'\, e^{-ik.x}\, e^{ik'.y}\, \langle 0|a_{\mathbf{k}}\, a_{\mathbf{k'}}^{\dagger}|0\rangle$$

$= \delta(k,k');$

$\langle 0|0\rangle = 1$

$$A_1 = \sum N^2\, e^{-ik.(x-y)}$$

$$\mathbf{A_1} = 1/V \sum_{\mathbf{k}} e^{-ik.(x-y)} / 2\omega$$

$\omega = \sqrt{(k^2+\mu^2)}$

Similarly,

$$\mathbf{A_2} = \langle 0|\varphi^{\dagger}(\boldsymbol{y})\,\varphi(\boldsymbol{x})\ |0\rangle$$
$$= \sum N \sum' N'\, \langle 0|(e^{-ik'.y}\, b_k)\,(e^{\mathbf{ik.x}}\, b_k^{\dagger})|0\rangle$$
$$= \sum N^2\, e^{+ik.(x-y)}$$

$$\mathbf{A_2} = 1/V \sum_{\mathbf{k}} e^{+ik.(x-y)} / 2\omega$$

Let $\xi^{\mu} = x^{\mu} - y^{\mu}$ *(This is the 4-vector.)*

Then, in the infinite volume limit,

$$i\,\Delta_F(\xi^{\mu}) = \theta(\xi^0) \int d^3k/(2\pi)^3\ \ e^{-ik.\xi}/2\omega$$
$$+\ \theta(-\xi^0) \int d^3k/(2\pi)^3\ \ e^{+ik.\xi}/2\omega$$

$$i\,\Delta_F(\xi^\mu) = \theta(\xi^0) \int d^3k/(2\pi)^3 \;\; e^{-ik.\xi} / 2\omega$$
$$+\;\theta(-\xi^0) \int d^3k/(2\pi)^3 \;\; e^{+ik.\xi} / 2\omega$$

Change **k** to −**k** in the second line

(this is the 3d vector)

$\Rightarrow$

$$i\,\Delta_F(\xi^\mu) = \int d^3k/(2\pi)^3 \;\; e^{i\,\mathbf{k}.\xi}$$
$$\{\, \theta(\xi^0)\exp(-i\,\omega\,\xi^0)/2\omega$$
$$+\;\theta(-\xi^0)\exp(i\,\omega\,\xi^0)/2\omega \,\}$$

$\omega = \sqrt{(k^2+\mu^2)}$

Theorem

$$\theta(\xi^0)\,\frac{e^{-i\omega\xi^0}}{2\omega} = \theta(\xi_0)\int_{-\infty}^{\infty} \frac{e^{-ik^0\xi^0}}{k_0^2-\omega^2+i\epsilon}\,\frac{i\,dk^0}{2\pi}$$

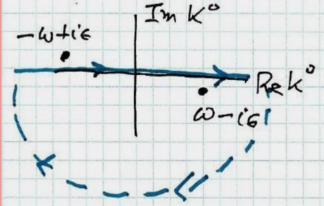
the limit ε→0 is understood

Proof

$$(k^0)^2-\omega^2+i\epsilon = (k^0-\omega+i\epsilon)(k^0+\omega-i\epsilon)$$

($\epsilon\to 0$ so $2\omega i\epsilon$ is same as $i\epsilon$)

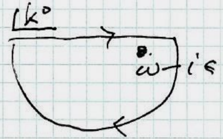
The complex k^0 plane



Factor $e^{-ik^0\xi^0}$ with $\xi^0>0$

⇒ close the contour BELOW;

$e^{-ik^0\xi^0}=0$ on the semicircle in the lower half plane, as $R\to\infty$



= by the RESIDUE THEOREM

$$= (-2\pi i)\,\frac{e^{-i(\omega-i\epsilon)\xi^0}}{\omega-i\epsilon+\omega-i\epsilon}\,\frac{i}{2\pi}$$

$$= \frac{e^{-i\omega\xi^0}}{2\omega} \text{ in limit } \epsilon\to 0 \qquad \text{Q.E.D.}$$

Theorem

$$\theta(\xi^0)\,\frac{e^{-i\omega\xi^0}}{2\omega} = \theta(\xi_0)\int_{-\infty}^{\infty}\frac{e^{-ik^0\xi^0}}{k_0^2-\omega^2+i\epsilon}\,\frac{i\,dk^0}{2\pi}$$

Similarly,

$$\theta(-\xi^0)\,\frac{e^{i\omega\xi^0}}{2\omega} = \theta(-\xi^0)\int_{-\infty}^{\infty}\frac{i\,dk^0}{2\pi}\,\frac{e^{-ik^0\xi^0}}{(k^0)^2-\omega^2+i\epsilon}$$

Close the contour in the upper half plane; get residue of the pole at $k^0 = -\omega + i\epsilon$.

I had before

$$\{\, \theta(\xi^0)\exp(-i\,\omega\,\xi^0)/2\omega + \theta(-\xi^0)\exp(i\,\omega\,\xi^0)/2\omega \,\}$$

$$= \int_{-\infty}^{\infty}\frac{i\,dk^0}{2\pi}\;\frac{e^{-i\,k0\;\xi0}}{k_0^{\;2}-\omega^2+i\varepsilon}$$

because $\theta(\xi^0)+\theta(-\xi^0) = 1$,

where $\omega = \sqrt{(k^i k^i + \mu^2)}$

Now go back to

$$i\,\Delta_F(\xi^\mu) = \int d^3k/(2\pi)^3 \quad e^{\,\boldsymbol{i}\,\mathbf{k}.\xi}$$

$$\{\, \theta(\xi^0)\, \exp(-\,\boldsymbol{i}\,\omega\,\xi^0)\,/2\omega$$

$$+\ \theta(-\xi^0)\, \exp(\,\boldsymbol{i}\,\omega\,\xi^0)\,/2\omega \,\}$$

$$\{\ \dots\ \} = \int_{-\infty}^{+\infty} \frac{i\,dk^0}{2\pi}\; \frac{e^{-i\,k0\,\xi0}}{(k^0)^2 - k^2 - \mu^2 + i\varepsilon}$$

FINAL RESULT

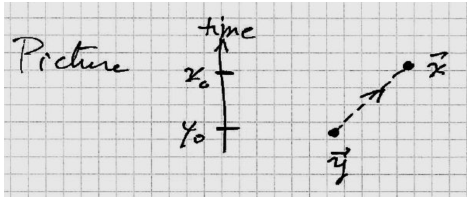
$$i\,\Delta_F(\xi) = i\int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik\cdot\xi}}{k^2-\mu^2+i\epsilon}$$

$$= \langle 0|\,T\,\phi(x)\,\phi^\dagger(y)\,|0\rangle$$

where $x^\mu - y^\mu = \xi^\mu$ (4 vectors)

INTERPRETATION of the PROPAGATOR

Loosely speaking, $\Delta_F(\,x^\mu - y^\mu\,)$ describes ***either*** propagation of a meson from spacetime point y^μ to spacetime point x^μ ,



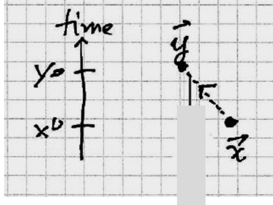
case $x^0 > y^0 \Rightarrow$

$\langle 0|\,\phi(x^\mu)\,\phi^\dagger(y^\mu)\,|0\rangle$

annihilation **a** at x

creation **a⁺** at y

or propagation of an antiparticle from x^μ to y^μ .



case $y^0 > x^0 \Rightarrow$

$\langle 0|\,\phi^\dagger(\vec{y})\,\phi(\vec{x})\,|0\rangle$

annihilation **b** at y

creation **b⁺** at x

... but more about this next time.

Pion exchange as a model for nucleon-nucleon scattering

pages 49 - 51

To illustrate how these propagators arise, we shall consider qualitatively nucleon–nucleon scattering. In this process there will be two nucleons but no mesons present in the initial and final states (i.e. before and after the scattering). The scattering, i.e. the interaction, corresponds to the exchange of virtual mesons between the nucleons. The simplest such process is the one-meson exchange, schematically illustrated in Fig. 3.3. The continuous lines represent the nucleons, the dashed lines the mesons. As before, two situations arise according to whether $t > t'$ or $t' > t$. In the actual calculation, all values of x and x' are integrated over, corresponding to emission and absorption of the meson occurring at any two space–time points.

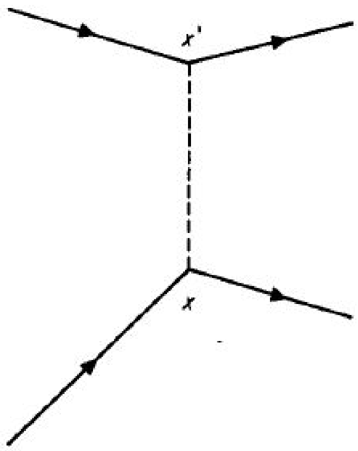


Fig. 3.4. Feynman graph for the one-meson contribution to nucleon–nucleon scattering.

To get the complete formula for the transition amplitude, we'll need the material in Chapters 6 and 7 (mainly PHY 955).

Homework Problems due Friday Feb 17

Problem 21.
Mandl and Shaw problem 3.3.

Problem 22.
Mandl and Shaw problem 3.4.

Problem 23.
Mandl and Shaw problem 3.5.

Problem 24.
a problem on the Dirac equation