MANDL AND SHAW

CHAPTER 4 THE DIRAC FIELD

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PROBLEMS; 4.1 4.2 4.3 4.4 4.5

APPENDIX A THE DIRAC EQUATION

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PROBLEMS; A.1 A.2

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Notations, conventions and units

- Section 1.2: RATIONALIZED gaussian electromagnetic units
- Section 2.1: Relativity notations
- Section 6.1: Natural units

$x^{\mu} = (x^{0}, x^{1}, x^{2}, x^{3})$; $x^{0} = c\, t$

$g^{\mu\nu} = g_{\mu\nu} = \textbf{DIAG}(1 , -1, -1, -1)$

$x_{\mu} = g_{\mu\nu}\, x^{\nu}$; $(x_{0} , x_{i}) = (x^{0} , x^{i})$

$p.x = g_{\mu\nu}\, p^{\mu} x^{\nu} = p^{0} x^{0} - p^{i} x^{i}$

$p^{2} = p.p = \mathrm{E}^{2} - \mathrm{p}^{2} = \mathrm{m}^{2}$

$\partial_{\mu} = \partial /\partial x^{\mu} = (\partial /\partial x^{0} , \partial /\partial x^{i})$

$\hbar = 1$ and $c = 1$

$\alpha = e^{2} /(4\pi\hbar c) = 1 / 137.037$

The Dirac Equation
(Read Appendix A and Sec 4.2)

/1/ Recall the Schroedinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^{2}}{2m} \nabla^{2} \psi$$

The plane wave solutions are

$$\Psi(\boldsymbol{x},t) = C\, e^{\, i (\mathbf{p}.\boldsymbol{x} - E\, t)} \qquad (\hbar = 1)$$

$$H\, \Psi = E\, \Psi \quad \Rightarrow \quad E = p^{2} /2m$$

(nonrelativisic)

$$\mathbf{P} = - i\, \nabla, \quad \text{so} \quad \mathbf{P}\, \Psi = \mathbf{p}\, \Psi(\boldsymbol{x},t)$$

The plane wave is an eigenstate of momentum.

/2/ The Dirac equation

We need an equation with these properties:

(i) linear in time, (unlike Klein Gordon)

(ii) with plane wave solutions,

(iii) with $E = \sqrt{p^2+m^2}$.

$$\Psi(x,t) \propto e^{\,i(\mathbf{p.x} - E\,t)} = e^{-i\,p.x}$$

$$i\,\partial\Psi/\partial t = H\,\Psi$$

Should we try

$$H\,\Psi = \sqrt{p^2+m^2}\;\Psi\,;$$

i.e., $H = \sqrt{P^2+m^2}$?

But that is a nonlocal operator.

- To be consistent with relativity, t and (x, y, z) should be treated similarly; because the Lorentz transformations mix t and (x, y, z).

So let's try

$$i\,\partial\Psi/\partial t = (\boldsymbol{\alpha}.\mathbf{P} + \beta\, m)\,\Psi$$

such that

sqrt of P^2+m^2

$$(\boldsymbol{\alpha}.\mathbf{P} + \beta m)^2 = P^2 + m^2$$

The quantities β and $(\alpha_x, \alpha_y, \alpha_z)$ will be ***matrices***.

Now try

$$\Psi(\boldsymbol{x},t) \propto e^{\,i(\mathbf{p.x} - E\,t)}\;u\,.$$

$$(\vec{\alpha}\cdot\vec{p} + \beta m)\, u = E\, u$$

$$(\vec{\alpha}\cdot\vec{p} + \beta m)^2 u = E^2 u$$

$$\left.\begin{aligned} &\alpha^i \alpha^j p^i p^j + \beta^2 m^2 \\ &+ 2 m p^i (\alpha^i \beta + \beta \alpha^i) \end{aligned}\right\} = \vec{p}^{\,2} + m^2$$

So we must require

$$\tfrac{1}{2}(\alpha^i \alpha^j + \alpha^j \alpha^i) = \delta_{ij}$$

$$\alpha^i \beta + \beta \alpha^i = 0$$

$$\beta^2 = 1$$

β and (α_x , α_y , α_z)

Since they don't commute, they must be matrices.

<u>Four - vector notations</u> (Appendix A)

Define $\gamma^0 = \beta$;

also, $(\gamma^1, \gamma^2, \gamma^3) = (\beta\alpha_x, \beta\alpha_y, \beta\alpha_z)$

UPPER AND LOWER INDICES:

$\{x^0, x^1, x^2, x^3\} = \{ct, x, y, z\}$ (c = 1)

$\{x_0, x_1, x_2, x_3\} = \{ct, -x, -y, -z\}$

$g_{\mu\nu}$ = diag(1,–1,–1,–1)

$\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\} = \beta\{1, \alpha_x, \alpha_y, \alpha_z\}$

$\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\} = \beta\{1, -\alpha_x, -\alpha_y, -\alpha_z\}$

$\gamma \cdot A = \gamma^\mu A_\mu = \gamma_\mu A^\mu = \gamma^0 A^0 - \gamma^i A^i$

$$i\frac{\partial \psi}{\partial t} = -i\vec{\alpha}\cdot\nabla\psi + \beta m \psi$$

$$i\gamma^0 \frac{\partial \psi}{\partial t} = -i\vec{\gamma}\cdot\nabla\psi + m\psi$$

$$i\left(\gamma^0 \frac{\partial}{\partial x^0} + \vec{\gamma}\cdot\nabla\right)\psi - m\psi = 0$$

$$\boxed{i\,(\gamma^0\,\partial_0 + \gamma^i\,\partial_i\,)\,\psi - m\,\psi = 0}$$

That is the Dirac equation.

Various notations may be used

$$i\,\gamma^\mu\,\partial_\mu\psi - m\,\psi = 0$$

$$i\,\gamma \bullet \partial\psi - m\,\psi = 0$$

$$i\,\not{\partial}\,\psi - m\,\psi = 0$$

Slash notation

$$\not{Q} = \gamma^\mu\,Q_\mu\,.$$

/3/ The gamma matrices

What are the gamma matrices?

They are not unique.

The gamma matrices are 4 X 4 matrices, defined by certain anticommutation relations:

$$\{\,\alpha^i\,,\,\alpha^j\,\} = 2\,\delta_{ij}$$

$$\{\,\gamma^i\,,\,\gamma^j\,\} = \{\,\beta\alpha^i\,,\,\beta\alpha^j\,\} = -\,2\,\delta_{ij}$$

$$\{\,\gamma^i\,,\,\gamma^0\,\} = \{\,\beta\alpha^i\,,\,\beta\,\} = 0$$

$$\{\,\gamma^0\,,\,\gamma^0\,\} = 2\,(\gamma^0)^2 = 2$$

Thus, the defining equation is

$$\{\,\gamma^\mu\,,\,\gamma^\nu\,\} = 2\,g^{\mu\nu} \qquad (\bigstar)$$

Theorem. If $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and U is a unitary matrix ($U^\dagger U = 1$), then $\{\gamma'^\mu, \gamma'^\nu\} = 2g^{\mu\nu}$ where $\gamma'^\mu = U\gamma^\mu U^\dagger$.

Proof.

$$
\begin{aligned}
\{\gamma'^\mu, \gamma'^\nu\} &= U\gamma^\mu U^\dagger U \gamma^\nu U^\dagger \\
&\quad + U\gamma^\nu U^\dagger U\gamma^\mu U^\dagger \\
&= U\{\gamma^\mu, \gamma^\nu\}U^\dagger \\
&= 2g^{\mu\nu}UU^\dagger = 2g^{\mu\nu}
\end{aligned}
$$

Exercise.

Find U such that $\gamma_M^\mu = U\gamma^\mu U^\dagger$.

- *The standard representation* ("Dirac rep.") *for the gamma matrices is*

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \text{ and } \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

where $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and σ_i = i'th Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Exercise. Verify (★).
(We never raise the index on a Pauli matrix!)

- *The Majorana representation*

$$\gamma_M^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \text{ and } \gamma_M^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix} \text{ etc}$$

See (A.79)

which is sometimes convenient.

- (Peskin and Schroeder use yet a different representation.)

For most calculations, we don't need to use any specific representation of the gamma matrices. Instead we can use some identities that are true for all representations.

$$\{ \gamma^\mu , \gamma^\nu \} = 2\, g^{\mu\nu} \qquad (\bigstar)$$

/4/ Examples of gamma matrix identities

▌ Trace ($\gamma^\mu \gamma^\nu$)

Lemma. Trace(BA) = Trace(AB).

Proof. Trace(BA)

$$= \sum B_{rs} A_{sr} = \sum A_{sr} B_{rs}$$

$$= \text{Trace(AB)}.$$

Even if A and B do not commute, i.e., $BA \neq AB$, always Tr(BA) = Tr(AB).

$$\mathrm{Tr}\, \gamma^\mu \gamma^\nu = \tfrac{1}{2} \mathrm{Tr} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$$

$$= \tfrac{1}{2} \mathrm{Tr}\; 2 g^{\mu\nu} \mathbb{1}_{4\times4} = 4 g^{\mu\nu}$$

▌ Trace ($\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$)

$$\mathrm{Tr}\, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$$

$$= \mathrm{Tr}\, \{\gamma^\mu, \gamma^\nu\} \gamma^\rho \gamma^\sigma - \mathrm{Tr}\, \gamma^\nu \gamma^\mu \gamma^\rho \gamma^\sigma$$

$$= 2 g^{\mu\nu} \cdot 4 g^{\rho\sigma} - \mathrm{Tr}\, \gamma^\nu \{\gamma^\mu, \gamma^\rho\} \gamma^\sigma$$

$$+ \mathrm{Tr}\, \gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma$$

$$= 8 g^{\mu\nu} g^{\rho\sigma} - 8 g^{\mu\rho} g^{\nu\sigma}$$

$$+ \mathrm{Tr} \left[\gamma^\nu \gamma^\rho \{\gamma^\mu, \gamma^\sigma\} \right] - \mathrm{Tr}\, \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu$$

$$= 8 g^{\mu\nu} g^{\rho\sigma} - 8 g^{\mu\rho} g^{\nu\sigma} + 8 g^{\mu\sigma} g^{\nu\rho}$$

$$- \mathrm{Tr}\, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \quad \leftarrow \text{by the Lemma}$$

$$\mathrm{Tr}\, \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4 g^{\mu\nu} g^{\rho\sigma} - 4 g^{\mu\rho} g^{\nu\sigma}$$

$$+ 4 g^{\mu\sigma} g^{\nu\rho}$$

- $\gamma^\mu \gamma^\rho \gamma_\mu$

$$
\begin{aligned}
\gamma^\mu \gamma^\rho \gamma_\mu &= \{\gamma^\mu, \gamma^\rho\} \gamma_\mu - \gamma^\rho \gamma^\mu \gamma_\mu \\
&= 2 g^{\mu\rho} \gamma_\mu - \gamma^\rho 4 \\
&= -2 \gamma^\rho
\end{aligned}
$$

- Etc.

We’ll use many such identities.
See the Appendix, Sections A.2 and A.3.

/5/ The Dirac spinors

- Plane wave solutions of the Dirac equation

$$(i\not\partial - m)\psi = 0$$

$$\psi_\alpha(x) = \underbrace{e^{i(\vec{p}\cdot\vec{x} - Et)}}_{e^{-ip\cdot x}}\, u_\alpha(\vec{p}, \lambda) \quad \leftarrow \boxed{(\alpha = 1, \ldots, 4)}$$

spin index

$$p\cdot x = p_\mu x^\mu$$
$$= p^0 x^0 - \vec{p}\cdot\vec{x}$$
$$= Et - \vec{p}\cdot\vec{x}$$

$$[i\gamma^\mu(-ip_\mu) - m]\, u = 0$$

$$(\not p - m)\, u = 0 \qquad \text{i.e. } \not p_{\alpha\beta}\, u_\beta - m u_\alpha = 0$$

u_α is an eigenvector of $\not p$

Now, consider $(\not p - m)(\not p + m)$

$$= \not p \not p - m^2 = \gamma^\mu \gamma^\nu p_\mu p_\nu - m^2$$
$$= \tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\} p_\mu p_\nu - m^2 = g^{\mu\nu} p_\mu p_\nu - m^2$$
$$= p^2 - m^2 = 0$$

Therefore, $\underline{u(\vec{p}, \lambda)}$ can be any of the 4 columns of $\not p + m$.

- In the standard (Dirac) representation:

$$\not{p} + m = \gamma^0 E - \gamma^i p^i + m$$

$$= \left[\begin{array}{c|c} E+m & -\vec{\sigma}\cdot\vec{p} \\ \hline \vec{\sigma}\cdot\vec{p} & -E+m \end{array}\right] \quad \text{in 2x2 block form}$$

The first 2 columns are

$$\begin{bmatrix} E+m & 0 \\ 0 & E+m \\ p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{bmatrix}$$

Thus the particle spinors are

$$u(\vec{p},1) = N \begin{pmatrix} E+m \\ 0 \\ p^3 \\ p^1 + ip^2 \end{pmatrix} \quad \text{and} \quad u(\vec{p},2) = N \begin{pmatrix} 0 \\ E+m \\ p^1 - ip^2 \\ -p^3 \end{pmatrix}$$

The other 2 solutions are antiparticle spinors (homework)

Normalization choice

This can be done in different ways.
We'll follow Mandl and Shaw ;
eq. (A.27) ;

$$u_r^\dagger(\vec{p})\, u_s(\vec{p}) = \frac{E}{m}\,\delta_{rs} \qquad r, s \in \{1, 2\}$$

$$v_r^\dagger(\vec{p})\, v_s(\vec{p}) = \frac{E}{m}\,\delta_{rs} \qquad r, s \in \{1, 2\}$$

Also, define

$$\bar{u} = u^\dagger \gamma^0 \quad \text{and} \quad \bar{v} = v^\dagger \gamma^0$$

Then $\quad \bar{u}_r(\vec{p})\, u_s(\vec{p}) = \delta_{rs} \quad \text{and} \quad \bar{v}_r(\vec{p})\, v_s(\vec{p}) = -\delta_{rs}$

and

$$\sum_{r=1}^{2} \left(u_r \bar{u}_r - v_r \bar{v}_r \right) = \mathbb{1}_{4\times 4}$$

completeness

Check the normalization:

$$u_1 = N \begin{bmatrix} E+m \\ 0 \\ p_z \\ p_x + ip_y \end{bmatrix} \quad \text{and} \quad u_1^\dagger = N \begin{bmatrix} E+m & 0 & p_z & p_x - ip_y \end{bmatrix}$$

$$u_1^\dagger u_1 = N^2 \{ (E+m)^2 + p_z^2 + p_x^2 + p_y^2 \}$$

$$= N^2 \{ E^2 + 2Em + m^2 + \vec{p}^{\,2} \}$$

$$= N^2 \, 2E(E+m)$$

$$= \frac{E}{m} \quad \Rightarrow \quad N = \frac{1}{\sqrt{2m(E+m)}}$$

Now

$$\bar{u}_1 u_1 = u_1^\dagger \gamma^0 u_1 = u_1^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u_1$$

$$= N^2 \{ (E+m)^2 - p_z^2 - p_x^2 - p_y^2 \}$$

$$= N^2 \{ E^2 + 2Em + m^2 - \vec{p}^{\,2} \}$$

$$= N^2 \, 2m(E+m) = 1 \checkmark$$

"ENERGY PROJECTION OPERATORS" ;
Section A.5 ;

$$\Lambda^\pm(\vec{p}) = \begin{cases} \sum_r u_r(\vec{p}) \, \bar{u}_r(\vec{p}) \\ -\sum_r v_r(\vec{p}) \, \bar{v}_r(\vec{p}) \end{cases}$$

these are 4x4 matrices!

$$\Lambda^\pm(\vec{p}) = \frac{\pm \not{p} + m}{2m}$$

Homework Problems due Friday, Feb. 17

Problem 24.

A. Determine the Dirac spinors $v_1(p)$ and $v_2(p)$ for antiparticles.

B. Determine the polarization sum $\Lambda^{-}(p)$ for antiparticles.