MANDL AND SHAW

CHAPTER 4 The Dirac Field

4.1 The number representation for fermions ✔

4.2 The Dirac equation ✔

4.3 Second quantization

4.4 The fermion propagator

4.5 The electromagnetic interaction and gauge invariance

PROBLEMS; 4.1 4.2 4.3 4.4 4.5

APPENDIX A The Dirac Equation

A1 A2 A3 A4 ✔

A5 A6 **A7** **A8**

PROBLEMS; A.1 A.2

Quantization of the Dirac Field (Section 4.3)

First quantization, for the Dirac equation,

$$(i\gamma\cdot\partial - m)\,\psi = 0\,.$$

The plane wave solutions are

$$e^{-ip.x}\,u_s(p) \qquad s \in \{1,2\}$$

$p.x = p^0x^0 - \mathbf{p.x}$

$p^0 = \sqrt{\mathbf{p}^2+m^2} \equiv E_p$

(positive energy solutions)

$$e^{+ip.x}\,v_s(p)$$

(negative energy solutions)

Spinor definitions :

$$(\gamma\cdot p - m)\,u_s(p) = 0 \qquad s \in \{1,2\}$$

$$(\gamma\cdot p + m)\,v_s(p) = 0$$

$\bar{u}\,u = 1$ and $\bar{v}\,v = -1$ where $\bar{u} = u^{\dagger}\gamma^0$.

Now, we can expand $\psi(x)$ in plane waves,

$$\psi(x) = \sum_{\vec{p}s}\left(\frac{m}{\Omega E_p}\right)^{1/2}\left\{ c_s(\vec{p})\,u_s(\vec{p})\,e^{-ip\cdot x} + d_s^{\dagger}(\vec{p})\,v_s(\vec{p})\,e^{ip\cdot x}\right\}$$

and

$$\bar{\psi}(x) = \sum_{\vec{p}s}\left(\frac{m}{\Omega E_p}\right)^{1/2}\left\{ c_s^{\dagger}(\vec{p})\,\bar{u}_s(\vec{p})\,e^{ip\cdot x} + d_s(\vec{p})\,\bar{v}_s(\vec{p})\,e^{-ip\cdot x}\right\}$$

The coefficients $c_s(\mathbf{p})$ and $d_s(\mathbf{p})$ will become annihilation operators;
$c_s^{\dagger}(\mathbf{p})$ and $d_s^{\dagger}(\mathbf{p})$ creation operators.

① Second quantization -- creation and annihilation operators

(familiar from many particle systems)

$$\{c_r(\vec{p}), c_s^+(\vec{p}')\} = \delta_{rs}\,\delta_{\vec{p}\vec{p}'}$$

$$\{d_r(\vec{p}), d_s^+(\vec{p}')\} = \delta_{rs}\,\delta_{\vec{p}\vec{p}'}$$

all other anticommutators are 0

E.g., $\{c_r, c_s\} = 0$; $\{c_r, d_s^+\} = 0$; etc

② The equal time anticommutation relations (E.T.aC.R.)

- $\{\psi_\alpha(x), \psi_\beta(y)\}$ with $x^0 = y^0$

$\propto$ $\{$ c and d^+ , c and d^+ $\}$

$= 0$

- $\{\psi_\alpha(x), \psi_\beta^+(y)\}$ with $x^0 = y^0$

$$= \sum_{\vec{p}s}\sum_{\vec{p}'s'} \frac{m}{\Omega\sqrt{EE'}} \{ c_s(\vec{p})\, u_{s\alpha}(\vec{p})\, e^{-ip\cdot x} + d_s^+(\vec{p})\, v_{s\alpha}(\vec{p})\, e^{ip\cdot x},\; c_{s'}^+(\vec{p}')\, \bar{u}_{s'\beta'}(\vec{p}')\, e^{ip'\cdot y} + d_{s'}(\vec{p}')\, \bar{v}_{s'\beta'}(\vec{p}')\, e^{-ip'\cdot y} \} (\gamma^0)_{\beta'\beta}$$

$\{c_s(p), c_{s'}^+(p')\} = \delta_{ss'}\,\delta_{\vec{p}\vec{p}'}$

$\{d_s^+(p), d_{s'}(p')\} = \delta_{ss'}\,\delta_{\vec{p}\vec{p}'}$

$\Rightarrow E = E'$ so $e^{-ip\cdot x} e^{ip'\cdot y} = e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$

$e^{ip\cdot x} e^{-ip'\cdot y} = e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}$

$$= \sum_{\vec{p}s} \frac{m}{\Omega E} \{ e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\, u_{s\alpha}(p)\, \bar{u}_{s\beta'}(p) + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\, v_{s\alpha}(p)\, \bar{v}_{s\beta'}(p) \} (\gamma^0)_{\beta'\beta}$$

$$= \sum_{\vec{p}} \frac{m}{\Omega E} \{ e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\, \Lambda^+_{\alpha\beta'}(p) + e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\, (-)\, \Lambda^-_{\alpha\beta'}(p) \} (\gamma^0)_{\beta'\beta}$$

$$\Lambda^\pm(p) = \frac{\pm\not{p} + m}{2m} = \frac{\pm\gamma^0 E \mp \vec{\gamma}\cdot\vec{p} + m}{2m}$$

In the second term, let $\vec{p} \to -\vec{p}$ because we $\sum_{\vec{p}}$

$$= \sum_{\vec{p}} \frac{m}{\Omega E} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{1}{2m} \{ E - \vec{\gamma}\cdot\vec{p}\,\gamma^0 + m\gamma^0 + E + \vec{\gamma}\cdot\vec{p}\,\gamma^0 - m\gamma^0 \}_{\alpha\beta}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\, \mathbb{1}_{\alpha\beta} = \delta^3(\vec{x}-\vec{y})\, \mathbb{1}_{\alpha\beta} \quad (*)$$

③ *Canonical quantization*

/3a/ We start with a Lagrangian ...

$$L = \int \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi \; d^3x$$

Let's check the field equations:

$$\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta \dot{f}}\right) - \frac{\delta L}{\delta f} = 0$$

Variation of $\bar{\psi}$ $\Rightarrow$

$$0 - (i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0 \qquad \text{(Dirac equation)}$$

Variation of ψ $\Rightarrow$

$$\frac{\partial}{\partial t}\left(\bar{\psi}\, i\gamma^0\right) - \left(-i\,\nabla\bar{\psi}\cdot\vec{\gamma} - m\bar{\psi}\right) = 0$$

$$i\,\frac{\partial}{\partial x^\mu}\left(\bar{\psi}\gamma^\mu\right) + m\bar{\psi} = 0$$

↑ equivalent to the adjoint of the Dirac equation

tricky to prove!

/3b/ ...from which we can derive the commutation relation;
Dirac method of quantization;

$$L = \int \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi \, d^3x$$

The canonical momentum π conjugate to ψ

is $\pi = \frac{\delta L}{\delta \dot{\psi}} = \bar{\psi} i\gamma^0 = i\psi^+$

so the Dirac quantization rule is

$$\{ \psi(\vec{x}t), \pi(\vec{y}t) \} = i\, \delta^3(\vec{x}-\vec{y})$$

or $\{ \psi(\vec{x}t), \psi^+(\vec{y},t) \} = \delta^3(\vec{x}-\vec{y})$
which agrees with (*)

④ *The Feynman propagator* = $S_F(x-y)$ *(Section 4.4)*

Recall from Chap. 3 — we want the vacuum expectation value of the *time-ordered product of fields*.

(*Do you remember why?*)

- So here is the definition of $S_F(x-y)$:

$$i\, S_F(x-y) = \langle 0 \,|\, T\, \psi(x)\, \bar\psi(y) \,|\, 0 \rangle$$

$$i\, S_F = \langle 0|\, \psi(x)\, \bar\psi(y)\, |0\rangle \qquad \text{if } x^0 > y^0$$
$$i\, S_F = -\langle 0|\, \bar\psi(y)\, \psi(x)\, |0\rangle \qquad \text{if } x^0 < y^0$$

Here is the formula for $S_F(x-y)$, as a Fourier integral:

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{\not p + m}{p^2 - m^2 + i\epsilon}\, e^{-ip\cdot(x-y)}$$

$S_F(x-y)$ is "*the Green's function with Feynman boundary conditions*"; recall $\Delta_F(x-y)$.

⑤ *Derivation of the Fourier integral*

First, calculate $\langle 0|\, \psi(x)\, \bar\psi(y)\, |0\rangle$

$$= \sum_{\vec p s} \sum_{\vec p' s'} \frac{m}{\Omega\sqrt{EE'}}$$
$$\langle 0| \left[c_s(p)\, u_s(p)\, e^{-ip\cdot x} + d_s^\dagger(p)\, v_s(p)\, e^{ip\cdot x} \right]$$
$$\left[c_{s'}^\dagger(p')\, \bar u_{s'}(p')\, e^{ip'\cdot y} + d_{s'}(p')\, \bar v_{s'}(p')\, e^{-ip'\cdot y} \right] |0\rangle$$

There are 4 terms; 3 terms are 0; $\langle 0|\, c_s(p)\, c_{s'}^\dagger(p')\, |0\rangle = \delta_{ss'}\, \delta_{\vec p \vec p'}$;

$$= \sum_{\vec p s} \frac{m}{\Omega E}\, u_s(p)\, \bar u_s(p)\, e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{m}{E}\, \Lambda^+(p)\, e^{-ip\cdot(x-y)} \quad ; \quad \Lambda^+(p) = \frac{\not p + m}{2m}$$

$$= (i\not\partial + m)\; i\Delta^+(x-y) \qquad (\text{see Eq. } 3.39)$$

$$\Delta^+(\xi) = \frac{-i}{2} \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ik\cdot\xi}}{\sqrt{k^2+m^2}}$$

Also, similarly,

$$-\langle 0|\, \bar\psi(y)\, \psi(x) |0\rangle = (i\not\partial + m)\; i\Delta^+(y-x)$$

Thus,

$$i S_{F\,\alpha\beta}(x-y) = \theta(x^0-y^0)\,\langle 0|\,\psi_\alpha(x)\,\bar\psi_\beta(y)\,|0\rangle - \theta(y^0-x^0)\,\langle 0|\,\bar\psi_\beta(y)\,\psi_\alpha(x)\,|0\rangle$$

$$= \theta(x^0-y^0)\,(i\partial\!\!\!/+m)\, i\Delta^+(x-y) + \theta(y^0-x^0)\,(i\partial\!\!\!/+m)\, i\Delta^+(y-x)$$

$$\Delta^+(y-x) = -\Delta^-(x-y)$$

$$= i(i\partial\!\!\!/+m)\left[\theta(x^0-y^0)\,\Delta^+(x-y) - \theta(y^0-x^0)\,\Delta^-(x-y)\right]$$

$$= i\,(i\partial\!\!\!/+m)\,\Delta_F(x-y) \quad \text{Eq. (3.56a)}$$

$$= i\,(i\partial\!\!\!/+m)\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \quad \text{Eq. (3.58)}$$

$$= \int \frac{d^4p}{(2\pi)^4}\,\frac{p\!\!\!/+m}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)}$$ ← i.e., $(p\!\!\!/+m)_{\alpha\beta}$

Homework Problems due Friday Feb 24

Problem 25.
Mandl and Shaw problem 4.1.

Problem 26.
Mandl and Shaw problem 4.2.

Problem 27.
Mandl and Shaw problem 4.3.

Problem 28.
Mandl and Shaw problem 4.4.

Problem 29.
Mandl and Shaw problem 4.5.