MANDL AND SHAW

CHAPTER 4 The Dirac Field

4.1 The number representation for fermions ✔

4.2 The Dirac equation ✔

4.3 Second quantization ✔

4.4 The fermion propagator ✔

4.5 The electromagnetic interaction and gauge invariance **(TODAY)**

PROBLEMS; 4.1 4.2 4.3 4.4 4.5

APPENDIX A The Dirac Equation

A1 A2 A3 A4 ✔

A5 A6 **A7** **A8** ← **LORENTZ TRANSFORMATIONS (FRIDAY)**

PROBLEMS; *A.1* *A.2*

Section 4.5.

THE ELECTROMAGNETIC INTERACTION AND GAUGE INVARIANCE

To describe electromagnetic effects, introduce potentials …

$\Phi(\mathbf{x},t)$ = the scalar potential ;

$$\mathbf{E} = -\nabla\Phi - (1/c)\,\partial\mathbf{A}/\partial t$$

$\mathbf{A}(\mathbf{x},t)$ = the vector potential ; $\mathbf{B} = \nabla \times \mathbf{A}$

$$A^\mu = (\Phi, \mathbf{A})$$
$$A_\mu = (\Phi, -\mathbf{A})$$

"Minimal substitution"

Start with the free Lagrangian $L = \int \pounds(\psi, \partial_\mu\psi)\, d^3x$.

Replace $$i\hbar\frac{\partial}{\partial t} \rightarrow i\hbar\frac{\partial}{\partial t} - q\Phi$$

and $$-i\hbar\nabla \rightarrow -i\hbar\nabla - \frac{q}{c}\mathbf{A}$$

$\pounds(\psi, \partial_\mu\psi)$ becomes

$\pounds(\psi, \partial_\mu\psi + i q A_\mu \psi)$

($\hbar$ = 1 and c = 1)

- The Dirac equation

$$\{ i \gamma^\mu [\partial_\mu + i q A_\mu(x)] - m \} \psi(x) = 0$$

- The Lagrangian $L = \int \mathcal{L}\, d^3x$

$$\mathcal{L} = \overline{\psi(x)} \{ i \gamma^\mu [\partial_\mu + i q A_\mu(x)] - m \} \psi(x)$$

$$= \overline{\psi(x)} [i \gamma^\mu \partial_\mu - m] \psi(x) - q \overline{\psi(x)} \gamma^\mu \psi(x) A_\mu(x)$$

$$= \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{interaction}}$$

- The interaction

$$\mathcal{L}_{\text{interaction}} = - j^\mu(x) A_\mu(x)$$

where $j^\mu(x) = q \overline{\psi(x)} \gamma^\mu \psi(x)$

- The electric current density

$j^\mu(x) = q\,\overline{\psi(x)}\,\gamma^\mu\,\psi(x)$ = *the 4-vector bilinear*

$j^0(x) = q\,\psi^\dagger(x)\,\psi(x) = \rho(x)$ **charge density**

$\mathbf{j}(x) = q\,\psi^\dagger(x)\,\gamma^0\boldsymbol{\gamma}\,\psi(x)$ **current density**

(Recall $\beta\boldsymbol{\gamma} = \boldsymbol{\alpha}$ = "velocity matrix".)

Gauge transformations ; $x = (t, \mathbf{x})$ *and* $\lambda(t, \mathbf{x})$

- **GAUGE TRANSFORMATIONS OF THE E.M. FIELDS**
  - $\mathbf{A}(x) \rightarrow \mathbf{A}'(x) = \mathbf{A}(x) + \nabla\lambda(x)$

    which implies $\mathbf{B}'(x) = \mathbf{B}(x)$
  - $\Phi(x) \rightarrow \Phi'(x) = \Phi(x) - \partial\lambda(x)/\partial t$

    which implies $\mathbf{E}'(x) = \mathbf{E}(x)$

The **E** and **B** fields are gauge invariant;
but the potentials are not gauge invariant.

In covariant form,

- $A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \lambda$.

Remember, $\lambda = \lambda(\mathbf{x},t)$.

$$\mathbf{B} = \nabla \times \mathbf{A}$$
$$\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$$

- **GAUGE TRANSFORMATIONS OF THE DIRAC FIELD**

$$\psi(x) \rightarrow \psi'(x) = e^{\,i\,q\,\lambda(x)}\,\psi(x)$$

The *current* is gauge invariant.

$$j^{\mu\prime}(x) = q\,\overline{\psi'(x)}\,\gamma^{\mu}\,\psi'(x)$$

$$= q\,\overline{\psi(x)}\;e^{-i\,q\,\lambda(x)}\,\gamma^{\mu}\,e^{\,i\,q\,\lambda(x)}\,\psi(x)$$

$$= q\,\overline{\psi(x)}\,\gamma^{\mu}\,\psi(x) \;=\; j^{\mu}(x)\,.$$

What about the Lagrangian?

$$\pounds = \overline{\psi}(\mathbf{x})\{ i\gamma^{\mu} [\partial_{\mu} + i q A_{\mu}(\mathbf{x})] - m \}\psi(\mathbf{x})$$

Substitute the gauge transformations …

$$\psi'(x) = e^{\mathbf{i\,q\,\lambda(x)}}\ \psi(x)$$

$$\overline{\psi'}(x) = \overline{\psi}(x)\ e^{-\mathbf{i\,q\,\lambda(x)}}$$

$$\partial_{\mu}\psi'(x) = e^{\mathbf{i\,q\,\lambda}}\ \partial_{\mu}\psi(x) + i\,q\,(\partial_{\mu}\lambda)\ e^{\mathbf{i\,q\,\lambda}}\ \psi(x)$$

$$A'_{\mu}(x) = A_{\mu}(x) - \partial_{\mu}\lambda$$

Thus,

$$\boxed{\pounds' = \pounds}$$

Theorem: For every continuous symmetry there is a conserved current.

Corresponding to the gauge invariance of **£**,
the conserved current is the electric current $j^\mu(x)$.

<u>Proof</u>

$j^\mu(x) = q\, \bar{\psi}(x)\, \gamma^\mu\, \psi(x)$

$\partial_\mu j^\mu(x) = q\, (\partial_\mu \bar{\psi})\, \gamma^\mu\, \psi + q\, \bar{\psi}(x)\, \gamma^\mu\, (\partial_\mu \psi)$

Dirac equation: $i\, \gamma^\mu\, [\, \partial_\mu + i\, q\, A_\mu(x)\,]\, \psi(x) - m\, \psi(x) = 0$

$\gamma^\mu\, (\partial_\mu \psi) = -\, i\, m\, \psi - i\, q\, \gamma^\mu A_\mu\, \psi$

Adjoint equation: $(\partial_\mu \psi^\dagger)(\gamma^\mu)^\dagger = i\, m\, \psi^\dagger + i\, q\, \psi^\dagger\, (\gamma^\mu)^\dagger\, A_\mu$

$(\partial_\mu \psi^\dagger)\, \gamma^0\, \gamma^\mu\, \gamma^0 = i\, m\, \psi^\dagger + i\, q\, \psi^\dagger\, \gamma^0\, \gamma^\mu\, \gamma^0\, A_\mu$

$(\partial_\mu \bar{\psi})\, \gamma^\mu = i\, m\, \bar{\psi} + i\, q\, \bar{\psi}\, \gamma^\mu\, A_\mu$

- $(\gamma^\mu)^\dagger = \gamma^0\, \gamma^\mu\, \gamma^0$;
- equation A.6 ;
- because $\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta\, m$ must be hermitian!

QED

Thus, $\partial_\mu j^\mu(x) = q\, (i\, m\, \bar{\psi} + i\, q\, \bar{\psi}\, \gamma^\mu\, A_\mu)\psi + q\, \bar{\psi}\, (-\, i\, m\, \psi - i\, q\, \gamma^\mu\, A_\mu\, \psi) = 0.$