MANDL and SHAW

CHAPTER 4 The Dirac Field

4.1 The number representation for fermions ✔

4.2 The Dirac equation ✔

4.3 Second quantization ✔

4.4 The fermion propagator ✔

4.5 The electromagnetic interaction and gauge invariance ✔

PROBLEMS; 4.1 4.2 4.3 4.4 4.5

APPENDIX A The Dirac Equation

A1 A2 A3 A4 ✔

A5 A6 **A7** **A8**

LORENTZ TRANSFORMATIONS

PROBLEMS; *A.1* *A.2*

How is the Dirac equation consistent with special relativity? (Appendix, section A.7)

$$(i\ \gamma \cdot \partial - \mathrm{m})\ \psi(x) = 0$$

Now consider two inertial frames,

$x^{\mu} = \{ x^0, x^1, x^2, x^3 \}$ ref. frame $\mathfrak{F}$

$x'^{\mu} = \{ x'^0, x'^1, x'^2, x'^3 \}$ ref. frame $\mathfrak{F}'$

$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$ where $\Lambda^{\mu}{}_{\nu}$ = the Lorentz transformation matrix

Suppose $\psi(\boldsymbol{x})$ is a solution of the Dirac equation in the unprimed coordinates.

What is the solution in the primed coordinates?

LORENTZ TRANSFORMATIONS

$$x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

$$g_{\alpha\beta}\, \Lambda^{\alpha}{}_{\rho}\, \Lambda^{\beta}{}_{\sigma} = g_{\rho\sigma} \qquad (x'.x' = x.x)$$

Scalar fields

$$\varphi'(x') = \varphi(x)$$

Vector fields

$$A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu}\, A^{\nu}(x)$$

Spinor fields

$$\psi'_a(x') = S_{ab}\, \psi_b(x)$$

What is S_{ab} ?

Lorentz transformations of fields

The Lorentz transformation:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \qquad (\mu,\nu = 0\ 1\ 2\ 3)$$

sum over ν is implied.

For example, a boost in the z direction,

$$c\,t' = \gamma\,(c\,t - \beta\,z)$$

$$x' = x$$

$$y' = y$$

$$z' = \gamma\,(-\beta\,c\,t + z)$$

$$\beta = v/c \quad \text{and} \quad \gamma = (1 - \beta^2)^{-1/2}$$

A boost in the z direction,

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$$g_{\mu\nu}\,\Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}$$

{Comment: The inverse matrix is $\Lambda_{\nu\rho}$ because $(1-\beta^2)\gamma^2 = 1$}.

A field equation is covariant , i.e., consistent with special relativity, if the "same equation" applies for coordinates x^{μ} and x'^{μ} .

(Or … if the equation is written in tensors; here spinors.)

Example. The scalar field $\varphi(x)$ obeys this covariant equation

$$\partial_\mu \partial^\mu \varphi(x) + m^2 \varphi(x) = 0$$

($\hbar$ = 1 and c = 1)

The Lorentz transformation of the field is

$$\varphi(x) \rightarrow \varphi'(x') = \varphi(x) \, .$$

Proof $\quad x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$

$$\partial_\alpha \partial^\alpha = g^{\alpha\beta} \partial_\alpha \partial_\beta$$

$$= g^{\alpha\beta} \left(\frac{\partial x'^\rho}{\partial x^\alpha} \cdot \frac{\partial}{\partial x'^\rho} \right) \left(\frac{\partial x'^\sigma}{\partial x^\beta} \frac{\partial}{\partial x'^\sigma} \right)$$

$$= g^{\alpha\beta} \Lambda^\rho{}_\alpha \, \partial'_\rho \, \Lambda^\sigma{}_\beta \, \partial'_\sigma = g^{\rho\sigma} \partial'_\rho \partial'_\sigma$$

$$= \partial'_\rho \partial'^\rho$$

Therefore

$$\partial'_\mu \partial'^\mu \varphi'(x') = \partial_\mu \partial^\mu \varphi(x)$$

$$= - m^2 \varphi(x) \qquad = - m^2 \varphi'(x')$$

QED

Useful relations

$$x' \cdot x' = x \cdot x \quad \text{and} \quad \partial' \cdot \partial' = \partial \cdot \partial$$

$$g_{\mu\nu} \, \Lambda^\mu{}_\rho \, \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$

Example. The Dirac field $\psi(x)$

obeys this equation

$$i\,\gamma^\mu \partial_\mu\, \psi(x) - m\, \psi(x) = 0$$

($\hbar$ = 1 and c = 1)

- $\psi(x)$ is a 4 component function;

  $\psi_a(x)$ for $a = 1\ 2\ 3\ 4$;

  γ^μ is a 4 x 4 matrix ;

  $\psi_a(x)$ is called a *spinor* ;

  it is not a Lorentz vector ! ■

Be careful when we suppress the spinor index a.

e.g., $\gamma^\mu \partial_\mu\, \psi(x) = (\gamma^\mu)_{ab} \partial_\mu \psi_b(x)$

Now, *covariance* means

$$i\,\gamma^\mu \partial'_\mu\, \psi'(x') - m\, \psi'(x') = 0\ .$$

What is $\psi'(x')$?

Let's try $\psi'(x') = S\, \psi(x)$

where S is a 4 x 4 matrix; i.e.,

$$\psi'_a(x') = S_{ab}\, \psi_b(x)$$

(sum over b implied)

So

$$i\gamma^\mu \partial'_\mu S \psi(x) - m S \psi(x) = 0$$

$$i\gamma^\mu \partial'_\mu S\psi(x) - m S\psi(x) = 0$$

$$i S^{-1}\gamma^\mu S \underbrace{\partial'_\mu \psi(x)} - m\psi(x) = 0$$

$$\frac{\partial\psi}{\partial x'^\mu} = \frac{\partial\psi}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\mu} = \frac{\partial x^\alpha}{\partial x'^\mu}\partial_\alpha\psi$$

$$i S^{-1}\gamma^\mu S \frac{\partial x^\alpha}{\partial x'^\mu}\partial_\alpha\psi - m\psi(x) = 0$$

We have $i\gamma^\alpha\partial_\alpha\psi - m\psi = 0$,

so we require

$$S^{-1}\gamma^\mu S \frac{\partial x^\alpha}{\partial x'^\mu} = \gamma^\alpha$$

$$S^{-1}\gamma^\mu S \frac{\partial x^\alpha}{\partial x'^\mu} \times \frac{\partial x'^\rho}{\partial x^\alpha} = \gamma^\alpha \times \frac{\partial x'^\rho}{\partial x^\alpha}$$

- $x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow \dfrac{\partial x'^\rho}{\partial x^\alpha} = \Lambda^\rho{}_\alpha$

- $\dfrac{\partial x'^\rho}{\partial x^\alpha}\dfrac{\partial x^\alpha}{\partial x'^\mu} = \dfrac{\partial x'^\rho}{\partial x'^\mu} = \delta_{\rho\mu} = g^\rho{}_\mu$

$$S^{-1}\gamma^\rho S = \Lambda^\rho{}_\alpha\gamma^\alpha$$

Finally, what is $S(\Lambda)$?

$$S\ \gamma^\mu S^{-1} = \Lambda^\mu{}_\nu\gamma^\nu$$

$$\boldsymbol{\Lambda}^\mu{}_\nu = \begin{matrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{matrix}$$

OK, let's try

$$S = \exp\{\, i/4\ \omega_{\alpha\beta}\,\sigma^{\alpha\beta}\,\}$$

like spin operator

where $\sigma^{\alpha\beta} = i/2\,[\,\gamma^\alpha, \gamma^\beta\,]$;

and we'll guess

$$\omega_{\alpha\beta} = \omega\,(\,\delta_{\alpha0}\delta_{\beta3} - \delta_{\alpha3}\delta_{\beta0}\,)\,.$$

- $S = \exp\{\, i/4\ \omega_{\alpha\beta}\, \sigma^{\alpha\beta} \,\}$
- $\omega_{\alpha\beta} = \omega\, (\, \delta_{\alpha 0}\delta_{\beta 3} - \delta_{\alpha 3}\delta_{\beta 0}\,)$.
- $S = \exp\{\, i/4\ \omega\, (\, \sigma^{03} - \sigma^{30}\,)\, \}$

  $= \exp\{\, i/2\ \omega\, \sigma^{03} \,\}$
- $\sigma^{03} = i/2\, [\, \gamma^0 , \gamma^3\,]$

$= i/2 \times$

$$\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}\begin{vmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{vmatrix} - \begin{vmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

(2x2 blocks)

$$= i \begin{vmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{vmatrix}$$

- $S = \exp\left\{ -\tfrac{1}{2}\, \omega \begin{matrix} 0 & \sigma_z \\ \sigma_z & 0 \end{matrix} \right\}$
- $S = \begin{matrix} \cosh(\omega/2) & -\sigma_z \sinh(\omega/2) \\ -\sigma_z \sinh(\omega/2) & \cosh(\omega/2) \end{matrix}$

(2x2 blocks)

- Now demand

  $S\ \gamma^{\mu}\, S^{-1} = \Lambda^{\mu}{}_{\nu}\, \gamma^{\nu}$

❑ Now demand

$$S\ \gamma^{\mu}\ S^{-1} = \Lambda^{\mu}{}_{\nu}\ \gamma^{\nu}$$

❑ $S\,\gamma^{0}\,S^{-1} = \Lambda^{0}{}_{\nu}\,\gamma^{\nu}$

$= \gamma\,\gamma^{0} - \beta\,\gamma\,\gamma^{3} = \gamma \times \begin{pmatrix} 1 & -\beta\sigma_z \\ \beta\sigma_z & -1 \end{pmatrix}$

check

$$S\gamma^0 S^{-1} = \begin{bmatrix} \overset{(\omega/2)}{\cosh} & -\sigma_z\sinh \\ -\sigma_z\sinh & \cosh \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} \cosh & \sigma_z\sinh \\ \sigma_z\sinh & \cosh \end{bmatrix}$$

$$= \dots = \begin{bmatrix} \cosh^2+\sinh^2 & \sigma_z\, 2\sinh\cosh \\ -\sigma_z\, 2\sinh\cosh & -\cosh^2-\sinh^2 \end{bmatrix}$$

$$= \begin{bmatrix} \cosh\omega & \sigma_z\sinh\omega \\ -\sigma_z\sinh\omega & -\cosh\omega \end{bmatrix} = \begin{bmatrix} \gamma & -\sigma_z\beta\gamma \\ \sigma_z\beta\gamma & -\gamma \end{bmatrix}$$

So we require $\cosh\omega = \gamma$ and $\sinh\omega = -\beta\gamma$.

Note: $\cosh^2\omega - \sinh^2\omega = \gamma^2(1-\beta^2) = 1$ as it must be.

❑ $S\,\gamma^{1}\,S^{-1} = \Lambda^{1}{}_{\nu}\,\gamma^{\nu} = \gamma^{1} = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}$

check

$$S\gamma^1 S^{-1} = \begin{bmatrix} \overset{(\omega/2)}{\cosh} & -\sigma_z\sinh \\ -\sigma_z\sinh & \cosh \end{bmatrix} \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} \begin{pmatrix} \cosh & \sigma_z\sinh \\ \sigma_z\sinh & \cosh \end{pmatrix}$$

$$= \dots = \begin{pmatrix} 0 & \sigma_x(\cosh^2-\sinh^2) \\ -\sigma_x(\cosh^2-\sinh^2) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} = \gamma^1$$

❑ $S\,\gamma^{3}\,S^{-1} = \Lambda^{3}{}_{\nu}\,\gamma^{\nu}$

$= \gamma\,\gamma^{3} - \beta\,\gamma\,\gamma^{0} = \gamma \times \begin{pmatrix} -\beta & \sigma_z \\ -\sigma_z & \beta \end{pmatrix}$

check

$$S\gamma^3 S^{-1} = \begin{bmatrix} \sinh\omega & \sigma_z\cosh\omega \\ -\sigma_z\cosh\omega & -\sinh\omega \end{bmatrix} = \begin{bmatrix} -\beta\gamma & \sigma_z\gamma \\ -\sigma_z\gamma & \beta\gamma \end{bmatrix}$$

$$S = \begin{bmatrix} \cosh \omega/2 & -\sigma_z \sinh \omega/2 \\ -\sigma_z \sinh \omega/2 & \cosh \omega/2 \end{bmatrix} \quad \text{where } \cosh\omega = \gamma \text{ and } \sinh\omega = -\beta\gamma$$

<u>Example</u>

The spinors for a particle at rest are

$$u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Use the Lorentz transformation to determine the spinors for a particle with velocity $\mathbf{v} = v\,\mathbf{e}_z$.

$$S\,u_1 = \begin{pmatrix} \cosh\frac{\omega}{2} & +\sigma_z \sinh\frac{\omega}{2} \\ +\sigma_z \sinh\frac{\omega}{2} & \cosh\frac{\omega}{2} \end{pmatrix} \begin{pmatrix} \binom{1}{0} \\ \binom{0}{0} \end{pmatrix}$$

change the sign because the frame of reference moves with velocity $= -\vec{v}$.

$$= \begin{pmatrix} \binom{\cosh\frac{\omega}{2}}{0} \\ \binom{\sinh\frac{\omega}{2}}{0} \end{pmatrix}$$

$\cosh\omega = \gamma$

$= 2\cosh^2\frac{\omega}{2} - 1$

$\Downarrow$

$\cosh^2\frac{\omega}{2} = \frac{\gamma+1}{2} = \frac{E+mc^2}{2mc^2}$

$\sinh^2\frac{\omega}{2} = \frac{\gamma-1}{2} = \frac{E-mc^2}{2mc^2}$

$$S\,u_1 = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ \sqrt{(E-m)/(E+m)} \\ 0 \end{bmatrix} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ p_z/(E+m) \\ 0 \end{bmatrix}$$

$= u_1(\vec{p})$ where $\vec{p} = p_z\,\hat{e}_z$.

Similarly, $S\,u_2(\vec{p}=0) = u_2(\vec{p})$.

DIRAC FIELD BILINEARS

$\psi(x)$ is a spinor, under Lorentz transformations.

$\bar{\psi}\ \psi$ is a scalar

$\bar{\psi}\ \gamma^{\mu}\ \psi$ is a vector

$\bar{\psi}\ \gamma^{\mu}\ \gamma^{\nu}\ \psi$ is a tensor

$\bar{\psi}\ \gamma^{5}\ \psi$ is a pseudo-scalar

$\bar{\psi}\ \gamma^{\mu}\ \gamma^{5}\ \psi$ is a pseudo-vector

Proof : See Appendix Section A.7

$$\bar{\psi}'\psi' = \psi'^{\dagger}\gamma^0\psi'$$

$$= \psi^{\dagger} S(\Lambda)^{\dagger}\gamma^0 S(\Lambda)\psi$$

$$= \psi^{\dagger}\gamma^0\gamma^0 S(\Lambda)^{\dagger}\gamma^0 S(\Lambda)\psi$$

where

$$S(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

$$\gamma^0 S(\Lambda)^{\dagger}\gamma^0 = \gamma^0 e^{\frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^{\dagger}}\gamma^0$$

$$= e^{\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$$ because $\gamma^0(S^{\mu\nu})^{\dagger}\gamma^0 = S^{\mu\nu}$

$$\gamma^0 S(\Lambda)^{\dagger}\gamma^0 S(\Lambda) = e^{\frac{i}{2}\omega\cdot S}\, e^{-\frac{i}{2}\omega\cdot S} = 1$$

$$\bar{\psi}'\psi' = \bar{\psi}\psi$$ Q.E.D.

Etc, similarly.