Mandl and Shaw

Chapter 5. Photons: Covariant Theory

5.1. The classical field theory

5.2. Covariant quantization

5.3. The photon propagator

Problems; 5.1 5.2 5.3 5.4

Chapter 6. The S-Matrix Expansion

6.1. Natural Dimensions and Units

6.2. The S-matrix expansion

6.3. Wick's theorem

Problems; none

Read Section 6.1

- We usually do calculations with ħ = 1 and c = 1.
- So the final result will appear to have wrong units.
- But there will be a unique way to restore the factors of ħ and c to get the right units.
- *"restore the units" ; "units analysis" ; or "dimensional analysis".*

Read Section 6.1

Where are we going?

The Scattering Matrix **(S matrix)**

- Experiments in high-energy physics

are particle collisions. E.g., at the LHC,

$$P(p_1) + P(p_2) \longrightarrow F$$

- The final state $|F\rangle$

must have baryon number = 2, charge 2e, 4-momentum $p_1^\mu + p_2^\mu$, etc.

- The goal of the theory is to calculate the cross

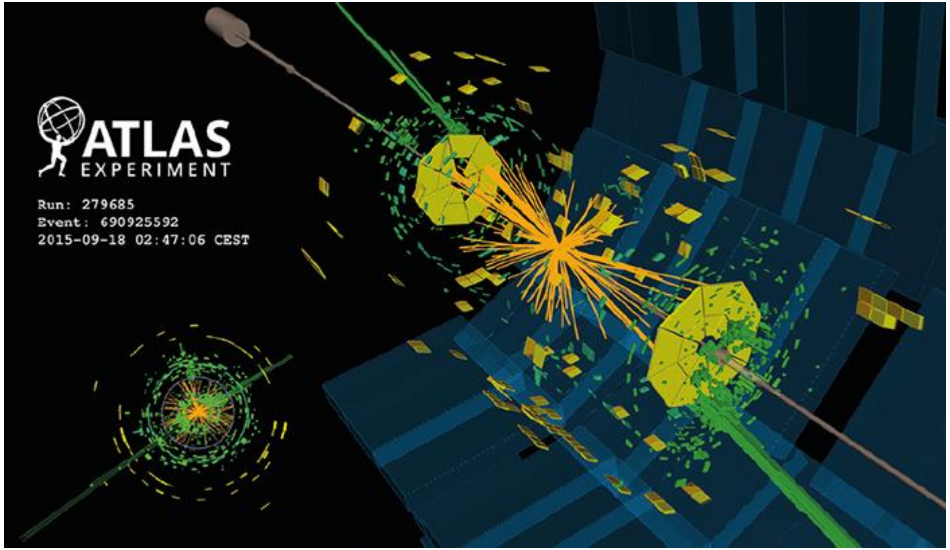
section for the process $P + P \longrightarrow F$:

$$rate = (inc.\ flux) \times |\langle F|S|I\rangle|^2$$

$$where \quad \langle F|S|I\rangle = \langle F, out\ |\ I, in\rangle$$

$$\langle\, F, out \mid I, in \,\rangle$$

The initial state consists 2 free particles.



The final state also consists of free particles; they are measured in the detector.

So we have ...

a free state
➡ interactions
➡ a different free state.

$\langle F, out \mid I, in \rangle$

The theory is a Lagrangian density

$$£ = £_{\text{free}} + £_{\text{int.}} \,.$$

We'll calculate the S-matrix, by treating $£_{\text{int}}$ *as a perturbation* of the free theory.

Adiabatic turning on. As a *technicality*, if we want to enforce the in/out states being free states, we could write

$$£_{\text{int.}} = £_{\text{int.}} \times f(t)$$

where $f(t) \to 0$ adiabatically as $t \to \pm\infty$.

How do we calculate time dependence in quantum theory?

Schroedinger picture

Heisenberg picture

But, to describe a small perturbation acting on a solvable system (like free fields) the best choice is to use the **Interaction picture**.

The interaction picture

Mandl and Shaw,
Section 1.5 and Section 6.2

Maiani and Benhar,
Chapter 10

The Interaction Picture

We use the *Interaction Picture* to calculate transition matrix elements in perturbation theory.

(1) Assume the Hamiltonian is

$$H = H_0 + H'$$

where the effects of H′ are small.

(2) The Scattering Matrix is

$$S = \hat{U}(\infty, -\infty)$$

where $\hat{U}(t_2, t_1)$ is the evolution operator from time t_1 to t_2.

The goal of quantum theory is to calculate *time evolution* of the states of the system.

Let $|\Psi\rangle$ be the state of the system at time t = 0.

Schroedinger picture

$$|\Psi,t\rangle_S = e^{-i Ht} |\Psi\rangle$$

Observable $A_s(t) = A$ *(independent of t)*

Heisenberg picture

$$|\Psi,t\rangle_H = |\Psi\rangle \text{ (independent of t)}$$

Observable $A_H(t) = e^{+i Ht} A e^{-i Ht}$

Interaction picture; suppose $H = H_0 + H'$

Observable $A_I(t) = e^{+i H_0 t} A e^{-i H_0 t}$

and

$$|\Psi,t\rangle_I = e^{+i H_0 t} e^{-i Ht} |\Psi\rangle = u(t) |\Psi\rangle$$

where

$$u(t) = e^{+i H_0 t} e^{-i Ht}$$

(unitary; S → I)

Proof

$$\langle \Psi,t|\, A \,|\Psi,t\rangle_S = \langle \Psi|\, e^{iHt} A e^{-iHt} \,|\Psi\rangle$$

$$= {}_I\langle \Psi,t|\, \underbrace{e^{iH_0t} e^{-iHt}}_{u}\, e^{iHt} A e^{-iHt}\, \underbrace{e^{iHt} e^{-iH_0t}}_{u^\dagger} \,|\Psi,t\rangle_I$$

$$= {}_I\langle \Psi,t|\, e^{iH_0t} A e^{-iH_0t} \,|\Psi,t\rangle_I$$

$$= \langle \Psi,t|\, A(t) \,|\Psi,t\rangle_I$$

Note

$$i\frac{\partial}{\partial t}|\Psi,t\rangle_I = i\frac{\partial u}{\partial t}|\Psi\rangle \quad \text{where} \quad u(t) = e^{iH_0t}e^{-iHt}$$

$$i\frac{\partial u}{\partial t} = e^{iH_0t}(-H_0 + H)e^{-iHt}$$

$$= e^{iH_0t}(H')e^{-iH_0t}u(t) = H_I'(t)u(t)$$

$$i\frac{\partial}{\partial t}|\Psi,t\rangle_I = H_I'(t)|\Psi,t\rangle_I$$

Result

In the Interaction Picture,

$$i\,\partial/\partial t\ |\Psi,t\rangle_I = H'_I(t)\ |\Psi,t\rangle_I$$

where

$$H'_I(t) = e^{+i H_0 t}\, H'\, e^{-i H_0 t}\ .$$

Description of a scattering process in the interaction picture.

As $t \to -\infty$, the state consists of free particles; then $H'_I(t) = 0$ and $|\Psi,t\rangle$ is constant. The state $|I\rangle$ consists of incoming particles *(which are known).*

As $t \to +\infty$, the state consists of free particles; then $H'_I(t) = 0$ and $|\Psi,t\rangle$ is constant. The state $|F\rangle$ consists of outgoing particles *(which are not known).*

The S matrix

Use the Interaction picture.

Suppose the state at time t_1 is $|\Psi, t_1\rangle = |I\rangle$, which consists of some specified incoming particles. (Eventually, let $t_1 \to -\infty$.)

At later times we have

$$|\Psi, t\rangle = e^{i H_0 (t - t_1)} e^{-i H (t - t_1)} |\Psi, t_1\rangle$$

Let t_2 be a long time after the collision; eventually let $t_2 \to +\infty$. The probability amplitude for the state to be $|F\rangle$ is $\langle F | \Psi, t_2\rangle$

$$= \langle F | e^{i H_0 (t_2 - t_1)} e^{-i H (t_2 - t_1)} |I\rangle$$

$$\langle F | \Psi, t_2\rangle = \langle F | u(t_2 - t_1) | I\rangle$$

$$= \langle F | S | I\rangle$$

in the *limit* $t_1 \to -\infty$ and $t_2 \to \infty$.

So,

$$S = \textit{lim}\ \hat{U}(t_2, t_1) = \textit{lim}\ u(t_2 - t_1)$$

<u>We have a differential equation to solve.</u>

We have $u(t) = e^{i H_0 \tau} e^{-i H \tau}$, and

$$i\, (\partial / \partial t)\, u(t) = H'_I(t)\, u(t) ,$$

i.e., $$i\, (\partial / \partial t)\, \hat{U}(t, t_1) = H'_I(t)\, \hat{U}(t, t_1) .$$

The initial condition is $\hat{U}(t_1, t_1) = 1$.

It looks like the solution might be an exponential.

$$i\,(\partial/\partial t)\,\hat{U}(t, t_1) = H'_I(t)\,\hat{U}(t, t_1) \text{ with } \hat{U}(t_1, t_1) = 1.$$

Now integrate from $t = t_1$ to t_2 ...

$$\hat{U}(t_2,t_1) = 1 + (-i)\int_{t_1}^{t_2} H_I'(t)\,\hat{U}(t,t_1)\,dt$$

$$= 1 + (-i)\int_{t_1}^{t_2} H_I'(t)\left\{1 + (-i)\int_{t_1}^{t} H_I'(t')\,\hat{U}(t',t_1)\,dt'\right\}dt$$

$$= 1 + (-i)\int_{t_1}^{t_2} H_I'(t)\,dt + (-i)^2\int_{t_1}^{t_2}dt\int_{t_1}^{t}dt'\,H_I'(t)\,H_I'(t')$$

$$+ (-i)^3\int_{t_1}^{t_2}dt\int_{t_1}^{t}dt'\int_{t_1}^{t'}dt''\,H_I'(t)\,H_I'(t')\,H_I'(t'')\,\hat{U}(t'',t_1)$$

= ..... = continue iterations

$$= 1 + \sum_{k=0}^{\infty}(-i)^{1+k}\int_{t_1}^{t_2}dt\int_{t_1}^{t}dt'\int_{t_1}^{t'}dt''\cdots\int_{t_1}^{t^{(k-1)}}dt^{(k)} \times H_I'(t)\,H_I'(t')\,H_I'(t'')\cdots H_I'(t^{(k)})$$

Change the notation for the integrated times:

$$\{t, t', t'', \ldots, t^{(k-1)}, t^{(k)}\} = \{s_1, s_2, s_3, \ldots, s_k, s_{k+1}\} \quad (\nu \equiv k+1)$$

$$\hat{U}(t_2,t_1) = 1 + \sum_{\nu=1}^{\infty}(-i)^{\nu}\int_{t_1}^{t_2}ds_1\int_{t_1}^{s_1}ds_2\int_{t_1}^{s_2}ds_3\cdots\int_{t_1}^{s_{\nu-1}}ds_\nu \times H_I'(s_1)\,H_I'(s_2)\,H_I'(s_3)\cdots H_I'(s_\nu)$$

Note that $s_1 > s_2 > s_3 > \ldots > s_\nu$;

so the product $H_I'(s_1)\cdots H_I'(s_\nu)$ is TIME ORDERED.

TRICK ~

$$\hat{U}(t_2,t_1) = 1 + \sum_{\nu=1}^{\infty}\frac{(-i)^{\nu}}{\nu!}\int_{t_1}^{t_2}ds_1\int_{t_1}^{t_2}ds_2\int_{t_1}^{t_2}ds_3\cdots\int_{t_1}^{t_2}ds_\nu \times T\{H_I'(s_1)\,H_I'(s_2)\cdots H_I'(s_\nu)\}$$

$$\hat{U}(t_2,t_1) = T\exp\left[-i\int_{t_1}^{t_2}ds\,H_I'(s)\right] \quad \text{TRICK}$$

Finally,

take the *limit* $t_1 \to -\infty$ and $t_2 \to \infty$;

$\Rightarrow \quad \hat{U}(\infty, -\infty) = S.$

The result is *Dyson's equation*.

$\Rightarrow$ Dyson's equation

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dt_1 \dots \int dt_n \; T\{ H'_I(t_1) \dots H'_I(t_n) \} .$$

i.e., $\int_{-\infty}^{\infty} dt_i$

Mandl and Shaw, Equation (6.22b);

Maiani and Benhar, Equation (10.38).

Now apply this result to Quantum Field Theory:

$$H'_I(t) = \int d^3x \; \mathfrak{H}'_I(\, t, \mathbf{x})$$

$\mathfrak{H}'$ = interaction energy *density*

$$S = \sum_{n=0}^{\infty} (-i)^n / n! \quad \int d^4x_1 \dots \int d^4x_n \; T\{ \mathfrak{H}'_I(x_1) \dots \mathfrak{H}'_I(x_n) \} .$$

Usually $\mathfrak{H}' = -\pounds_{\text{int.}}$

That is Mandl and Shaw Equation (6.23).

"This equation is *the Dyson expansion of the S-matri*x.
It forms the starting point for the approach to perturbation theory used in this book."