MANDL AND SHAW

Chapter 5. Photons: Covariant Theory

5.1. The classical field theory

5.2. Covariant quantization

5.3. The photon propagator

Problems; 5.1 5.2 5.3 5.4

Chapter 6. The S-Matrix Expansion

6.1. Natural Dimensions and Units ✔

6.2. The S-matrix expansion ✔

6.3. Wick's theorem

Problems; none

MAIANI AND BENHAR

Chapter 5 - quantization in the Coulomb gauge;

Section 5.2 - how the covariant propagator includes the Coulomb interactions;

Section 8.4 - the photon propagator for QED.

Section 5.1.
The Classical Electromagnetic Fields

Start with Maxwell's theory *in covariant form* ...

- The field tensor $F^{\mu\nu}(\boldsymbol{x})$

$$\begin{matrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{matrix}$$

$$F^{0i} = -F^{i0} = E_i$$
$$F^{ij} = \varepsilon_{ijk} B_k$$

- The field equations

(Cov. Maxwell equation #1)

$$\partial_\nu F^{\mu\nu} = s^\mu \; ;$$

$\nabla \bullet E = \rho$
$\nabla \times B = j$

which requires $\partial_\mu s^\mu = 0$;

conservation of charge

(Cov. Maxwell equation #2)

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \; .$$

$\nabla \cdot B = 0$ and $\nabla \times E = -\partial B / \partial t$

- The 4-vector potential $A^\mu(x)$

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$$

$F^{0i} = \partial^i A^0 - \partial^0 A^i$
$\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t$

$F^{ij} = \varepsilon_{ijk} B_k = \partial^j A^i - \partial^i A^j$
$= -\partial_j A^i + \partial_i A^j = \varepsilon_{ijk}(\nabla\times\mathbf{A})_k$

This satisfies (Cov.ME#2) automatically, i.e., for any $A^\mu(x)$.
And now (Cov.ME#1) becomes

$$\Box A^\mu - \partial^\mu (\partial_\nu A^\nu) = s^\mu \qquad (☆)$$

- The classical gauge transformation

$F^{\mu\nu}$ is invariant under the transformation

$$A^\mu(x) \longrightarrow A'^\mu(x) = A^\mu(x) + \partial^\mu f(x)$$

for any scalar function $f(x)$.

Proof:

$$F'^{\mu\nu} = \partial^\nu A'^\mu - \partial^\mu A'^\nu$$
$$= \partial^\nu (A^\mu + \partial^\mu f) - \partial^\mu (A^\nu + \partial^\nu f)$$
$$= F^{\mu\nu}$$

LAGRANGIAN DYNAMICS FOR ELECTROMAGNETISM

- What is the Lagrangian density?

Let's consider

$$£ = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - s_\mu A^\mu \qquad \text{(Lagr.1)}$$

Is it Lorentz invariant? **YES**

Is it gauge invariant? **NO**

But $(\delta £)_{G.T.} = \partial^\mu(-s_\mu f)$ so the action $A = \int £\, d^4x$, is gauge invariant. **OK**

The field equation for A^μ, i.e., Lagrange's equation, is …

$$\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \text{where } \phi = A_\rho$$

$$= \partial_\lambda \left(-\tfrac{1}{2} F^{\mu\nu} \underbrace{\frac{\partial F_{\mu\nu}}{\partial(\partial_\lambda A_\rho)}}_{= \delta_{\lambda\nu}\delta_{\rho\mu} - \delta_{\lambda\mu}\delta_{\rho\nu}} \right) + s^\rho$$

$$= -\partial_\nu F^{\rho\nu} + s^\rho = 0 \quad \text{(the Maxwell equation)}$$

$$\partial_\nu F^{\rho\nu} = s^\rho$$

However, this Lagrangian density (Lag.1) is not compatible with canonical quantization.

$$\pounds = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - s_\mu A^\mu \qquad \text{(Lag.1)}$$

Here is the problem: Try to calculate the canonical momentum as a 4-vector …

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad \text{and} \quad [\phi, \Pi_\phi] = i\hbar\, \delta^3(x-y)$$

$$\text{For } \phi = A_\rho \Rightarrow \pi^\rho = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\rho)} \quad (\rho = 0,1,2,3)$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = -\tfrac{1}{2} F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial(\partial_0 A_i)}$$

$$= -\tfrac{1}{2} F^{\mu\nu} \{ \delta_{0\nu}\delta_{i\mu} - \delta_{0\mu}\delta_{i\nu} \}$$

$$= -F^{i0} = E^i \quad \text{(that's OK)}$$

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0, \text{ so canonical quantization FAILS; } [A_0, \pi_0] \neq i\hbar\, \delta^3(\vec{x}-\vec{y}).$$

Another classical Lagrangian density (Fermi)

$$\pounds = -\tfrac{1}{2} (\partial_\nu A_\mu)(\partial^\nu A^\mu) - s_\mu A^\mu \qquad \text{(Lag.2)}$$

First check the field equation:

$$\partial_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\rho)} \right) - \frac{\partial \mathcal{L}}{\partial A_\rho} = \partial_\lambda \left(-\partial^\lambda A^\rho \right) + s^\rho$$

$$\Box A^\rho = s^\rho$$

It's OK, provided A^μ obeys the Lorentz gauge condition, $\partial_\rho A^\rho = 0$; *see (☆).*

Now check the compatibility with canonical quantization:

$$\pi^\rho = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\rho)} = -\partial^0 A^\rho \quad \text{not zero}$$

It's OK, but now there are 4 degrees of freedom: $Trans_1$, $Trans_2$, Long., A^0.

Long. and A^0 are unphysical.

The Lorentz gauge condition in the classical theory.

In the classical theory, suppose we have fields (i.e., the tensor $F^{\mu\nu}(x)$) with a 4-vector potential $A^{\mu}(x)$.

But suppose $\partial_{\mu}A^{\mu} \neq 0$.

There exists a gauge transformation $A^{\mu}(x) \longrightarrow A'^{\mu}(x) = A^{\mu}(x) + \partial^{\mu} f(x)$ such that $\partial_{\mu}A'^{\mu} = 0$.

Proof:

Let $f(x) = -\Box^{-1}(\partial_{\rho}A^{\rho})$

∴ The gauge transformation does not change the fields, $F^{\mu\nu}(x)$; and

$$\partial_{\mu}A'^{\mu} = 0.$$

The Lorentz gauge versus the Coulomb gauge.

$$\partial_{\mu}A^{\mu} = 0$$

versus

$$\nabla \cdot \mathbf{A} = 0 \text{ and } -\nabla^2\Phi = j_0$$

The theory is gauge invariant;
i.e., *the physical predictions are the same for either gauge condition.*

The Coulomb gauge has an advantage: it is a “unitary gauge”. But it has a disadvantage: it is not manifestly Lorentz invariant.

The Lorentz gauge has an advantage: it is manifestly Lorentz invariant. But is has a disadvantage: it has unphysical degrees of freedom.

Section 5.2: *① Use the “Gupta-Bleuler formalism” to impose the condition $\partial_{\mu}A^{\mu} = 0$; and ② then don’t worry about it.*

Chapter 13: Path Integrals
Section 13.4.Gauge Independent Quantization?

"… … …
As in the canonical formulation, the electromagnetic field cannot be consistently quantized using path integrals without 'fixing a gauge.'
"

Chapter 14: Quantum Chromodynamics
Section 14.1. Gluon Fields
Section 14.1.5.
The electromagnetic field revisited.

"… … …
It will be instructive to comment briefly on the result of applying the *Faddeev-Popov procedure* to the electromagnetic field…"
(In QED the ghost fields integrate to a mere normalization constant.)

But now going back to Section 5.1:

Plane wave solutions

The free field theory ($s^\mu(x) = 0$) has just

$$\Box A^\mu = 0 .$$

(impose Lorentz condition later)

The plane wave solutions are

$$A^\mu(x) = \varepsilon_r{}^\mu(\mathbf{k})\, e^{-i\,k.x} \quad \textit{with} \quad r \in \{ 0, 1, 2, 3 \}$$

where $k^0 = \pm|\mathbf{k}|$.

The ***four*** polarization vectors are normalized in some way; $\varepsilon_1{}^\mu$ and $\varepsilon_2{}^\mu$ are spatial and transverse w.r.t. **k**, $\varepsilon_3{}^\mu$ is spatial and longitudinal, and $\varepsilon_0{}^\mu$ is (1,0,0,0).

The *general* solution is

$$A^\mu(x) = \sum_{\vec{k}} \sum_{r=0}^{3} \left(\frac{\hbar c^2}{2\Omega\omega}\right)^{1/2} \epsilon_r^\mu(\vec{k}) \left\{ a_r(\vec{k})\, e^{-ik\cdot x} + a_r^*(\vec{k})\, e^{ik\cdot x} \right\}$$

where $\epsilon_0^\mu = (1,0,0,0)$ time like,

$\epsilon_3^\mu = \vec{k}/|\vec{k}|$ longitudinal, $\epsilon_{1,2}^\mu$ = transferse

Homework Problems

due Friday March 3

Problem 27
Mandl and Shaw problem 5.1

Problem 28
Mandl and Shaw problem 5.2

Problem 29
Mandl and Shaw problem 5.3

Problem 30
Mandl and Shaw problem 5.4