MANDL AND SHAW

SECTION 5.2.
COVARIANT QUANTIZATION

Review the covariant equations for the classical electromagnetic field

$$F = \textit{Lorentz curl } A \quad ; \quad F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$$

In the Lorentz gauge ($\partial_\mu A^\mu = 0$) we can use the Lagrangian density

$$L = -\tfrac{1}{2} (\partial_\nu A_\mu)(\partial^\nu A^\mu) - s_\mu A^\mu$$

Apply canonical quantization to this Lagrangian density. Then calculate

$$[\, A^\mu(x) \, , A^\nu(y) \,] = i \, D^{\mu\nu} (x - y)$$

and

$$\langle 0 | T \, A^\mu(x) \; A^\nu(y) \, | 0 \rangle = i \, D_F^{\mu\nu} (x - y)$$

$L = -\frac{1}{2} (\partial_\nu A_\mu)(\partial^\nu A^\mu) - s_\mu A^\mu$

Recall the real scalar field,

$$L_\varphi = \frac{1}{2}(\partial_\nu \varphi)(\partial^\nu \varphi) - \frac{1}{2} m^2 \varphi^2$$

$$[\varphi(x), \varphi(y)] = i\, \Delta(x-y)$$

$$\langle 0 |\, T\, \varphi(x)\, \varphi(y)\, | 0 \rangle = i\, \Delta_F(x-y)$$

Note that the free e.m. field has

$$L = -\frac{1}{2}(\partial_\nu A^0)(\partial^\nu A^0) + \frac{1}{2}(\partial_\nu A^i)(\partial^\nu A^i)$$

(sum i = 1 2 3)

- $A^i(x)$ is just like $\varphi(x)$ *with m = 0*

$\therefore \quad [A^i(x), A^i(y)] = i\, \Delta(x-y)$

$\therefore \quad \langle 0 |\, T\, A^i(x)\, A^i(y)\, | 0 \rangle = i\, \Delta_F(x-y)$

(no sum on i)

Different *i* and *j* are independent

$\therefore \quad [A^i(x), A^j(y)] = i\, \delta_{ij}\, \Delta(x-y)$

$\& \langle 0 |\, T\, A^i(x)\, A^j(y)\, | 0 \rangle = i\, \delta_{ij}\, \Delta_F(x-y)$

- $A^0(x)$ is a little different (the sign)

$$\Pi^0 = \partial L \,/\, \partial(\partial A^0/\partial t) = -\,\partial A^0/\partial t$$

(compare $\Pi_\varphi = \partial\varphi/\partial t$)

So the commutator is

$$[\, A^0(x), A^0(y)\,] = -i\, \Delta(x-y)$$

__Commutator result__

$$[\, A^\mu(x), A^\nu(y)\,] = -i\, g^{\mu\nu}\, \Delta(x-y)$$
$$= i\, D^{\mu\nu}(x-y)$$

$\therefore \quad D^{\mu\nu}(x-y) = -\, g^{\mu\nu}\, \Delta(x-y)$

↑ *__(with m = 0)__*

- As for the propagator,

The same argument implies

$$\langle 0 |\ T\, A^\mu(x)\, A^\nu(y)\, | 0 \rangle = -i\, g^{\mu\nu}\, \Delta_F(x-y)$$

$$= i\, D_F^{\mu\nu}(x-y)$$

$$D_F^{\mu\nu}(x-y) = -\, g^{\mu\nu}\, \Delta_F(x-y)$$

(with m = 0)

The Fourier integral,

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, D_F^{\mu\nu}(k)\, e^{-i\, k.(x-y)}$$

$$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + i\,\varepsilon}$$

Expansion in plane waves

$$A^\mu(x) = \sum_{\vec{k}} \sum_{r=0}^{3} \left(\frac{\hbar c^2}{2\omega\Omega}\right)^{1/2} \epsilon_r^\mu(\vec{k}) \left\{ a_r(\vec{k})\, e^{-ik\cdot x} + a_r^\dagger(\vec{k})\, e^{ik\cdot x} \right\}$$

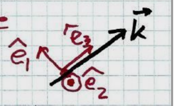
where $k^0 = |\vec{k}| = \omega_{\vec{k}}$.

The four-vector polarization vectors are defined like this:

$\epsilon_0^\mu(\vec{k}) = \eta^\mu = (1,0,0,0)$ timelike

$\epsilon_i^\mu(\vec{k}) = (0, \hat{e}_i(\vec{k}))$ for $i = 1\,2\,3$ spacelike

$\hat{e}_3(\vec{k}) = \frac{\vec{k}}{\omega}$ longitudinal



$\hat{e}_1, \hat{e}_2, \hat{e}_3$ form an orthogonal triad

($\hat{e}_1, \hat{e}_2$: transverse)

★The canonical commutation relation

$[\, A^\mu(x)\, ,\, A^\nu(y)\,] = -i\, g^{\mu\nu}\, \Delta(x-y)$ implies

$[a_r(\mathbf{k}), a^\dagger_s(\mathbf{k}')] = \delta_{rs}\, \delta_{\mathbf{kk}'}\, \zeta_r$ **where $\zeta_0 = -1$, $\zeta_i = +1$**

The constraint $\partial_\mu A^\mu = \mathbf{0}$; i.e., Lorentz gauge quantization

- We cannot set $\partial_\mu A^\mu = \mathbf{0}$ as an operator equation.

Proof:

$$[\partial_\mu A^\mu(x), A^\nu(y)] = -i\partial^\nu \Delta(x-y)$$

is not zero.

- The Gupta-Bleuler *formalism*:

(Comment: it's not a theory; it's not a model; it's a *formalism*.)

Apply the constraint, $\partial_\mu A^\mu = 0$, to the states of the Hilbert space; require

$$\partial_\mu A^{(+)\mu} |\Psi\rangle = 0 ,$$

for any physical state $|\Psi\rangle$.

Theorem

$$[\, a_3(\mathbf{k}) - a_0(\mathbf{k}) \,] \, |\Psi\rangle = 0 \quad \text{for all } \mathbf{k} .$$

Proof

$$\partial_\mu A^{(+)\mu} = \partial_\mu \sum_{\vec k} \sum_{r=0}^{3} \frac{1}{\sqrt{2\omega\Omega}} \epsilon_r^\mu(\vec k)\, a_r(\vec k)\, e^{-ik\cdot x}$$

$$= \sum_{\vec k} \sum_{r=0}^{3} \frac{1}{\sqrt{2\omega\Omega}} (-ik_\mu)\, \epsilon_r^\mu(\vec k)\, a_r(\vec k)\, e^{-ik\cdot x}$$

$$(-ik_\mu)\epsilon_r^\mu: \quad = k_0 \epsilon_r^0 - \vec k \cdot \hat\epsilon_r$$

$$= \omega\, \delta(r,0) - |\vec k|\, \delta(r,3) = \omega\big(\delta(r,0) - \delta(r,3)\big)$$

$$= \sum_{\vec k} \frac{\omega}{\sqrt{2\omega\Omega}} (-i)\big(a_0(\vec k) - a_3(\vec k)\big)\, e^{-ik\cdot x}$$

$\partial_\mu A^{(+)\mu}(x) |\Psi\rangle = 0$ for all x

implies $[a_0(\vec k) - a_3(\vec k)] |\Psi\rangle$ for all $\vec k$

(GB) $[\, a_3(\mathbf{k}) - a_0(\mathbf{k}) \,]\, |\Psi\rangle = 0,$

for any physical state $|\Psi\rangle$ and arb. $\mathbf{k}$.

What does it mean?

- Recall the free vacuum $|0\rangle$; it has

$$a_r(\mathbf{k})\, |0\rangle = 0 \quad \text{for} \quad r = 0\ 1\ 2\ 3$$

so $|0\rangle$ obeys **(GB).**

- Now consider $a^\dagger_r(\mathbf{k})\, |0\rangle$.

For r = 1, 2, it obeys **(GB);** creating any transverse photons will yield a physical state.

However, for r = 0 or 3, the state does not obey **(GB);** creating a single longitudinal photon or a single scalar photon will yield an unphysical state. *A physical state requires creating longitudinal and scalar photons together.*

- Example. Consider

$$|\psi\rangle = [\, a^\dagger_3(p) - a^\dagger_0(q) \,]\, |0\rangle$$

This state has

$$a_3(k)\, |\psi\rangle = \delta_{\mathbf{k},\mathbf{p}}\, |0\rangle$$
$$a_0(k)\, |\psi\rangle = \delta_{\mathbf{k},\mathbf{q}}\, |0\rangle$$

So $|\psi\rangle$ is a physical state (**GB**) provided that $\mathbf{p} = \mathbf{q}$.

(Mandl & Shaw, problem 5.2; homework)

- But that is just a gauge transformation.

(Mandl & Shaw, problem 5.3; homework)

Results of the formalism

In the Gupta-Bleuler formalism,

- we only calculate transition amplitudes between states with transverse photons;
- i.e., longitudinal and scalar photons are not included in asymptotic states;
- *but longitudinal and scalar photons do exist as virtual particles;*
- i.e., we use the full propagator

$$D_F^{\mu\nu}(k) = -g^{\mu\nu} / (k^2+i\varepsilon) .$$

This is all we need to proceed.

So now we could just forget about the Gupta-Bleuler formalism.

But note what Mandl and Shaw say at the end of Section 5.2:

"…
For most purposes, the complete formalism is not required.
[*footnote:* If interested, read 3 other books.]
"

An alternative approach (which is necessary for QCD but optional for QED) is to use ***functional integration*** with the ***Faddeev-Popov formalism***;
← gauge fixing and ghost fields.
(Mandl and Shaw, chapters 10 - 14)