Chapter 5. Photons: Covariant Theory
5.1. The classical field theory ✔
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5.3. The photon propagator
Problems; 5.1  5.2  5.3  5.4

Chapter 6. The S-Matrix Expansion
6.1. Natural Dimensions and Units ✔
6.2. The S-matrix expansion ✔
6.3. Wick’s theorem
Problems; none
SECTION 5.2.
COVARIANT QUANTIZATION

Review the covariant equations for the classical electromagnetic field

\[ F = \text{Lorentz curl } A \quad ; \quad F_{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu \]

In the Lorentz gauge \((\partial_\mu A^\mu = 0)\) we can use the Lagrangian density

\[ L = -\frac{1}{2} (\partial^\nu A^\mu)(\partial^\nu A^\mu) - s_\mu A^\mu \]

Apply canonical quantization to this Lagrangian density. Then calculate

\[ [ A^\mu(x), A^\nu(y) ] = i D^{\mu\nu}(x - y) \]

and

\[ \langle 0 | T A^\mu(x) A^\nu(y) | 0 \rangle = i D_F^{\mu\nu}(x - y) \]
Different \( i \) and \( j \) are independent

\[
[A^i(x) , A^j(y)] = i \delta_{ij} \Delta(x-y)
\]

\& \langle 0 | T A^i(x) A^j(y) | 0 \rangle = i \delta_{ij} \Delta_F(x-y)

\begin{itemize}
  \item \( A^0(x) \) is a little different (the sign)
  \[ \Pi^0 = \partial L / \partial (\partial A^0 / \partial t) = - \partial A^0 / \partial t \]
  \quad \text{(compare} \ \Pi_\phi = \partial \phi / \partial t)\]
\end{itemize}

So the commutator is

\[
[ A^0(x) , A^0(y) ] = -i \Delta(x - y)
\]

\[\textbf{Commutator result} \]

\[
[ A^\mu(x) , A^\nu(y) ] = -i g^{\mu\nu} \Delta(x-y)
\]

\[= i D^{\mu\nu} (x-y)\]

\[\therefore D^{\mu\nu} (x-y) = - g^{\mu\nu} \Delta(x-y)\]

\[\uparrow \quad \text{(with} \ m = 0)\]
As for the propagator,

The same argument implies

\[ \langle 0 | T A^\mu(x) A^\nu(y) | 0 \rangle = -i \ g^{\mu\nu} \Delta_F(x-y) \]

\[ = i \ D_F^{\mu\nu}(x-y) \]

\[ D_F^{\mu\nu}(x-y) = - g^{\mu\nu} \Delta_F(x-y) \]

\[ \text{(with } m = 0 \text{)} \]

The Fourier integral,

\[ D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \ D_F^{\mu\nu}(k) \ e^{-i \cdot k \cdot (x-y)} \]

\[ D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + i \epsilon} \]

Expansion in plane waves

The four-vector polarization vectors are defined like this:

\[ \epsilon^\mu_0(k) = (1, 0, 0, 0) \text{ timelike} \]

\[ \epsilon^\mu_1(k) = (0, \widehat{e}_2(k)) \text{ for } t = \text{ 1 or } 2 \text{ spacelike} \]

\[ \widehat{e}_3(k) = \frac{k}{|k|} \text{ longitudinal} \]

\[ \widehat{e}_1, \widehat{e}_2, \widehat{e}_3 \text{ form an orthogonal triad} \]

★The canonical commutation relation

\[ [ A^\mu(x), A^\nu(y) ] = -i \ g^{\mu\nu} \Delta(x-y) \] implies

\[ [a_r(k), a_s^\dagger(k')] = \delta_{rs} \delta_{kk'} \zeta_r \text{ where } \zeta_0 = -1, \ \zeta_i = +1 \]
Theorem \[ [ a_3(k) - a_0(k) ] |\Psi\rangle = 0 \quad \text{for all } k. \]

Proof

We cannot set \( \partial_{\mu} A^\mu = 0 \) as an operator equation.

Proof:

The Gupta-Bleuler formalism:

(Comment: it’s not a theory; it’s not a model; it’s a formalism.)

Apply the constraint, \( \partial_{\mu} A^\mu = 0 \), to the states of the Hilbert space; require

\[ \partial_{\mu} A^{(+)}_{\mu} |\Psi\rangle = 0, \]

for any physical state |\Psi\rangle.
Example. Consider

\[ |\psi\rangle = [ a^\dagger_3(p) - a^\dagger_0(q) ] |0\rangle \]

This state has

\[ a_3(k) |\psi\rangle = \delta_{k,p} |0\rangle \]
\[ a_0(k) |\psi\rangle = \delta_{k,q} |0\rangle \]

So \(|\psi\rangle\) is a physical state (GB) provided that \(p = q\).

(Mandl & Shaw, problem 5.2; homework)

But that is just a gauge transformation.

(Mandl & Shaw, problem 5.3; homework)

What does it mean?

- Recall the free vacuum \(|0\rangle\); it has

\[ a_r(k) |0\rangle = 0 \quad \text{for} \quad r = 0 1 2 3 \]

so \(|0\rangle\) obeys (GB).

- Now consider a \(\mathfrak{f}_r(k) |0\rangle\).

For \(r = 1, 2\), it obeys (GB); creating any transverse photons will yield a physical state.

However, for \(r = 0\) or 3, the state does not obey (GB); creating a single longitudinal photon or a single scalar photon will yield an unphysical state. *A physical state requires creating longitudinal and scalar photons together.*
This is all we need to proceed.
So now we could just forget about the Gupta-Bleuler formalism.
But note what Mandl and Shaw say at the end of Section 5.2:
“...
For most purposes, the complete formalism is not required.
[footnote: If interested, read 3 other books.]
”

An alternative approach (which is necessary for QCD but optional for QED) is to use functional integration with the Faddeev-Popov formalism;
\[ D_F^{\mu\nu}(k) = -g^{\mu\nu} / (k^2 + i\varepsilon) . \]