MANDL AND SHAW

Chapter 5. Photons: Covariant Theory

5.1. The classical field theory ✔

5.2. Covariant quantization ✔

5.3. The photon propagator

Problems; 5.1 5.2 5.3 5.4

Chapter 6. The S-Matrix Expansion

6.1. Natural Dimensions and Units ✔

6.2. The S-matrix expansion ✔

6.3. Wick's theorem

Problems; none

SECTION 5.3: THE PHOTON PROPAGATOR

(We could skip this, but there is something interesting here; interesting for the theory, but not really useful for applications of the theory.)

The Lorentz gauge ($\partial_\mu A^\mu = 0$) has a covariant photon propagator

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} D_F^{\mu\nu}(k)\, e^{-ik\cdot(x-y)}$$

$$D_F^{\mu\nu}(k) = \frac{-g_{\mu\nu}}{k^2+i\epsilon}$$

But it comes from the Gupta-Bleuler formalism, which seems abstract and unphysical.

The Coulomb gauge …

$$(\nabla \cdot \mathbf{A} = 0 \text{ and } \Phi = -\nabla^{-2} j^0)$$

… does not have unphysical “longitudinal and scalar photons”; but it is hard to use because the propagator is complicated and non-covariant.

Are the two gauge choices (Lorentz and Coulomb) consistent with each other?

Yes, because the physical predictions are the same for the two choices.

How can that be?
The propagators are very different. But propagators are not gauge invariant.
All physical predictions are gauge invariant.

Prove that they are ***gauge equivalent***

The crucial equation is equation (5.40), which is just a mathematical identity satisfied by the Lorentz gauge propagator…

$$D_F^{\mu\nu}(k) = \frac{1}{k^2+i\epsilon} \sum_{r=0}^{3} \epsilon_r^\mu(\vec{k})\, \epsilon_r^\nu(\vec{k})\, \zeta_r$$

$$(\zeta_0 = -1,\ \zeta_{123} = +1)$$

$$= \frac{1}{k^2+i\epsilon} \Big\{ \sum_{r=1}^{2} \epsilon_r^\mu(\vec{k})\, \epsilon_r^\nu(\vec{k})$$

TRANSVERSE POLARIZATIONS

$$+ \frac{(k^\mu - k\cdot n\, n^\mu)(k^\nu - k\cdot n\, n^\nu)}{(k\cdot n)^2 - k^2}$$

LONGITUDINAL POLARIZATION
[0 if $\mu=0$ or $\nu=0$]
[k^i if $\mu=i$]

$$- n^\mu n^\nu \Big\}$$

TIMELIKE
$n^\mu = (1,0,0,0)$

$$= {}_T D_F^{\mu\nu} + {}_C D_F^{\mu\nu} + {}_R D_F^{\mu\nu}$$

T : TRANSVERSE
C : COULOMB INTERACTION
R : REMAINDER

where …

I'm defining ...

$$ {}_C D_F^{\mu\nu}(k) \text{ is } \frac{n^\mu n^\nu}{(k\cdot n)^2 - k^2} = \frac{\delta_{\mu 0}\delta_{\nu 0}}{\vec{k}^2} $$

This is the Fourier transform of the instantaneous Coulomb interaction.

Proof

$$ \int \frac{d^4k}{(2\pi)^4}\, {}_C D_F^{\mu\nu}(k)\, e^{-ik\cdot(x-y)} $$

$$ = \int \frac{dk^0}{2\pi} e^{-ik^0(x^0-y^0)} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \frac{\delta_{\mu 0}\delta_{\nu 0}}{\vec{k}^2} $$

$$ = \delta(x^0-y^0)\, \frac{\delta_{\mu 0}\delta_{\nu 0}}{4\pi|\vec{x}-\vec{y}|} $$

And then the remainder is ...

$$ {}_R D_F^{\mu\nu}(k) \text{ is } \left\{ \frac{(k^\mu - k^0 n^\mu)(k^\nu - k^0 n^\nu)}{\vec{k}^2} - n^\mu n^\nu \right\} \frac{1}{k^2} $$

$$ - \frac{n^\mu n^\nu}{\vec{k}^2} \quad \text{(the remainder)} $$

$$ = \frac{1}{\vec{k}^2 k^2} \Big\{ k^\mu k^\nu - k^0 (n^\mu k^\nu + n^\nu k^\mu) $$

$$ + n^\mu n^\nu \big[\underbrace{(k^0)^2 - \vec{k}^2 - k^2}_{=0} \big] \Big\} $$

$$ = \frac{1}{\vec{k}^2 k^2} \left\{ k^\mu k^\nu - k^0 (n^\mu k^\nu + n^\nu k^\mu) \right\} $$

The remainder ${}_R D_F^{\mu\nu}(x - y)$ is not zero.

However, any transition matrix element that depends on $D_F^{\mu\nu}$ will involve the operator

$$\int d^4x \int d^4y \; s_{1\mu}(x)\, D_F^{\mu\nu}(x - y)\, s_{2\nu}(y)$$

where $\partial_\mu s_1^\mu = 0$ and $\partial_\mu s_2^\mu = 0$ *(conservation of charge).*

And ….

$$\int d^4x \, d^4y \; s_{1\mu}(x) \boxed{{}_R D_F^{\mu\nu}}(x-y)\, s_{2\nu}(y)$$

$$= \int d^4x \, d^4y \; s_{1\mu}(x) \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \underbrace{\left[k^\mu k^\nu d_1 + (n^\mu k^\nu + n^\nu k^\mu) d_2 \right]}_{{}_R D_F^{\mu\nu}(k)} s_{2\nu}(y)$$

$$= \int \frac{d^4k}{(2\pi)^4} \hat{s}_{1\mu}^{*}(k) \left[k^\mu k^\nu d_1 + (n^\mu k^\nu + n^\nu k^\mu) d_2 \right] \hat{s}_{2\nu}(k)$$

$\partial_\mu s_1^\mu(x) = 0$ implies $k_\mu \hat{s}_1^\mu(k) = 0$

and similarly $k_\mu \hat{s}_2^\mu(k) = 0$.

$= 0.$

So all physical contributions of ${}_R D_F^{\mu\nu}$ are 0.

We say "${}_R D_F^{\mu\nu}$ is gauge equivalent to 0".

MANDL AND SHAW

SECTION 7.1
FEYNMAN DIAGRAMS IN COORDINATE SPACE

To calculate transition probabilities, we need the S-matrix,

$$S_{FI} = \langle F |\ T \exp i \int d^4x\ \pounds_{\mathbf{int.}}(x)\ | I \rangle$$

where $| I \rangle$ and $| F \rangle$ are suitably normalized free particle states.

I.e., $$S_{FI} = \delta_{FI} + \sum_n S_{\mathbf{FI}}^{\mathbf{(n)}}$$

QED

$$\pounds = \pounds_{\psi} + \pounds_{A} + \pounds_{int}$$

$$\pounds_{\psi} = \bar{\psi}\,(i\gamma\cdot\partial - m)\,\psi$$

$$\pounds_{A} = -\tfrac{1}{2}\,(\partial_{\nu} A_{\mu})(\partial^{\nu} A^{\mu}) \qquad (\text{w/ } \partial_{\mu} A^{\mu}=0)$$

$$\pounds_{int} = e\;\mathbf{N}\{\,\bar{\psi}\,\gamma_{\mu}\,\psi\,A^{\mu}\,\}$$

(normal ordered)

$$\mathcal{L}_{\psi} + \mathcal{L}_{int} = \bar{\psi}\,(\underbrace{i\not\partial + e\not A}_{\text{"minimal coupling"}} - m)\,\psi$$

$$\mathcal{L}_{A} + \mathcal{L}_{int.} = -\frac{1}{2}(\partial_{\nu}A_{\mu})(\partial^{\nu}A^{\mu}) - s_{\mu}A^{\mu}$$

$$s_{\mu} = -e\,\bar{\psi}\gamma_{\mu}\psi = \text{current density}$$

Consider a second-order contribution to S_{FI}

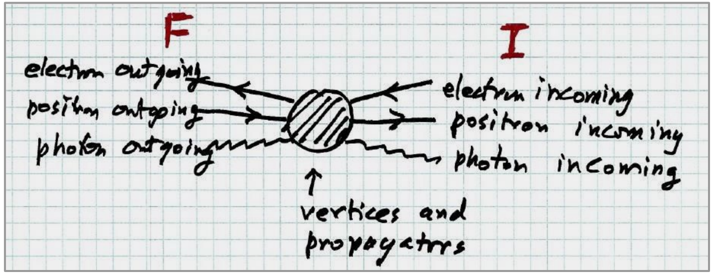
$$S^{(2)}_{FI} = \frac{i^2}{2!}\iint \langle F|\,T\,\mathcal{L}_I(x)\,\mathcal{L}_I(y)\,|I\rangle\, d^4x\, d^4y$$

$$= -\frac{e^2}{2}\iint \langle F|\,T :\bar{\psi}\gamma_{\mu}\psi A^{\mu}(x)::\bar{\psi}\gamma_{\nu}\psi A^{\nu}(y):|I\rangle\; d^4x\, d^4y$$

$$:ABC\ldots: \;=\; N(ABC\ldots)$$

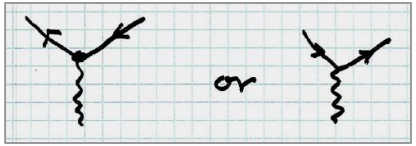
A Feynman diagram in configuration space consists of :

- vertices;
- external electron lines and internal electron lines;
- external photon lines and internal photon lines.



- ***Vertices:***

Associated factor
$= i\,e\,\gamma_\mu$



- ***External electron lines:***

Suppose $|I\rangle$ has an electron e($\mathbf{p}$,λ).
That must be annihilated by either $\psi(x)$ or $\psi(y)$.

$$\psi(x) = \sum_{\vec{p}} \sum_{\lambda} \left(\frac{2m}{2E\Omega}\right)^{1/2} \left\{ c_\lambda(\vec{p})\, u_\lambda(\vec{p})\, e^{-ip\cdot x} + d_\lambda^+(\vec{p})\, v_\lambda(\vec{p})\, e^{ip\cdot x} \right\}$$

So the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} u_\lambda(\vec{p})\, e^{-ip\cdot x} \quad \text{or} \quad e^{-ip\cdot y}$$

For a positron ***in*** $|F\rangle$ the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} v_\lambda(\vec{p}\,')\, e^{ip'\cdot x} \quad \text{or} \quad e^{ip'\cdot y}$$

$$\langle F|\, T\; ie\bar{\psi}\gamma_\mu \psi A^\mu(x)\; ie\bar{\psi}\gamma_\nu \psi A^\nu(y)\, |I\rangle$$

Suppose $|F\rangle$ has an electron e($\mathbf{p}'$,λ').
That must be created by either $\bar{\psi}(x)$ or $\bar{\psi}(y)$.

$$\bar{\psi}(x) = \sum_{\vec{p}} \sum_{\lambda} \left(\frac{2m}{2\Omega E}\right)^{1/2} \left[c_\lambda^+(\vec{p})\, \bar{u}_\lambda(\vec{p})\, e^{ip\cdot x} + d_\lambda(\vec{p})\, \bar{v}_\lambda(\vec{p})\, e^{-ip\cdot x} \right]$$

So the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} \bar{u}_{\lambda'}(\vec{p}\,')\, e^{ip'\cdot x} \quad \text{or} \quad e^{ip'\cdot y}$$

For a positron ***in*** $|I\rangle$ the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} \bar{v}_\lambda(\vec{p})\, e^{-ip\cdot x} \quad \text{or} \quad e^{-ip\cdot y}$$

▶ *External photon lines:*

Suppose |I⟩ has a photon $\gamma(\mathbf{k},r)$;
r = 1 or 2 only .
That must be annihilated by either
$A^\mu(\boldsymbol{x})$ or $A^\nu(\boldsymbol{y})$.

$$A^\mu(x) = \sum_{\vec{k}} \sum_{r=0}^{3} \left(\frac{1}{2\omega\Omega}\right)^{1/2} \epsilon_r^\mu(\vec{k}) \left\{ a_r(\vec{k})\, e^{-ik\cdot x} + a_r^\dagger(\vec{k})\, e^{ik\cdot x} \right\}$$

So the associated factor is

$$\frac{1}{\sqrt{2\Omega\omega}}\, \epsilon_r^\mu(\vec{k})\, e^{-ik\cdot x} \quad \text{or} \quad \epsilon^\nu e^{-ik\cdot y}$$

Suppose |F⟩ has a photon $\gamma(\mathbf{k}',r')$;
r′ = 1 or 2 only.
That must be created by either
$A^\mu(\boldsymbol{x})$ or $A^\nu(\boldsymbol{y})$.

Then the associated factor is

$$\frac{1}{\sqrt{2\Omega\omega'}}\, \epsilon_{r'}^\mu(\vec{k}')\, e^{ik'\cdot x} \quad \text{or} \quad \epsilon^\nu e^{ik'\cdot y}$$

An incoming line has a factor of exp(-iq.x) and and outgoing line has a factor of exp(+iq.x), where $\hbar q^\mu$ is the 4-momentum.

The other fields produce propagators.
(Wick's theorem)

▸ *Internal electron lines:*

Suppose Wick's theorem requires the contraction $\psi(x)\ \bar{\psi}(y)$ ...

Then the associated factor is

$$S_F(x - y) =$$

$$= (2\pi)^{-4} \int d^4p\ \ S_F(p)\ \ e^{-ip.(x-y)}$$

▸ *Internal photon lines:*

Suppose Wick's theorem requires the contraction $A^\mu(x)\ A^\nu(y)$ ...

Then the associated factor is

$$D_F^{\mu\nu}(x - y) =$$

$$= (2\pi)^{-4} \int d^4k\ \ D_F^{\mu\nu}(k)\ \ e^{-ik.(x-y)}$$