MANDL AND SHAW

Chapter 5. Photons: Covariant Theory ✔

5.1. The classical field theory ✔

5.2. Covariant quantization ✔

5.3. The photon propagator ✔

Problems; 5.1 5.2 5.3 5.4

Chapter 6. The S-Matrix Expansion

6.1. Natural dimensions and units ✔

6.2. The S-matrix expansion ✔

6.3. Wick's theorem

Problems; none

Chapter 7. Feynman Diagrams

7.1. F. diagrams in configuration space

7.2. F. diagrams in momentum space

SECTION 7.1
FEYNMAN DIAGRAMS IN COORDINATE SPACE

To calculate transition probabilities, we need the S-matrix,

$$S_{FI} = \langle F |\, T \exp i \int d^4x \;\mathcal{L}_{\mathbf{int.}}(x)\; | I \rangle$$

where $|I\rangle$ and $|F\rangle$ are suitably normalized free particle states.

I.e., $$S_{FI} = \delta_{FI} + \sum_{n} S^{(\mathbf{n})}{}_{\mathbf{FI}}$$

review

review

QED

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_{int}$$

$$\mathcal{L}_\psi = \bar{\psi}\,(i\gamma\cdot\partial - m)\,\psi$$

$$\mathcal{L}_A = -\tfrac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) \qquad (\text{w}/\ \partial_\mu A^\mu = 0)$$

$$\mathcal{L}_{int} = e\ N\{\ \bar{\psi}\,\gamma_\mu\,\psi\,A^\mu\ \}$$

(normal ordered)

$$\mathcal{L}_\psi + \mathcal{L}_{int} = \bar{\psi}\,(i\not{\partial} + e\not{A} - m)\,\psi$$

"minimal coupling"

$$\mathcal{L}_A + \mathcal{L}_{int} = -\tfrac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu) - S_\mu A^\mu$$

$S_\mu = -e\,\bar{\psi}\,\gamma_\mu\,\psi$ = current density

Consider a second-order contribution to S_{FI}

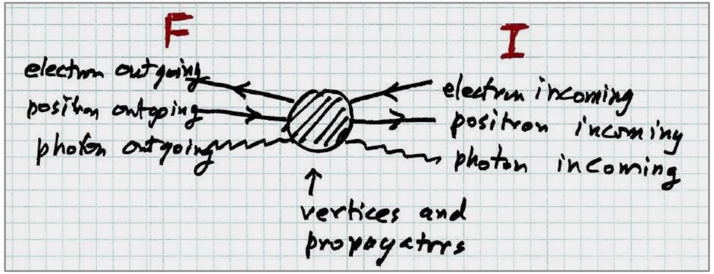
$$S^{(2)}_{FI} = \frac{i^2}{2!}\iint \langle F|\,T\,\mathcal{L}_I(x)\,\mathcal{L}_I(y)\,|I\rangle\, d^4x\, d^4y$$

$$= \frac{-e^2}{2}\iint \langle F|\,T\, :\bar{\psi}\gamma_\mu\psi A^\mu(x):\,:\bar{\psi}\gamma_\nu\psi A^\nu(y):\,|I\rangle\ d^4x\, d^4y$$

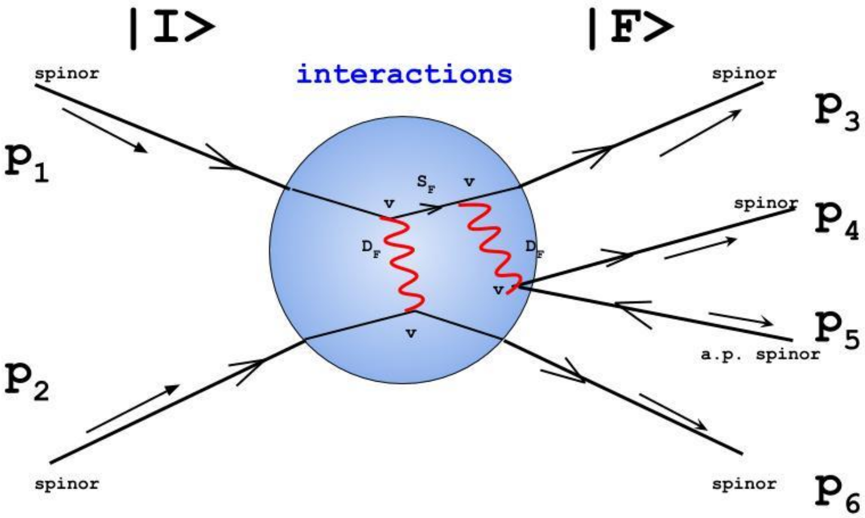
$:ABC\ldots: \;=\; N(ABC\ldots)$

A Feynman diagram in configuration space consists of :

- vertices;
- external electron lines and internal electron lines;
- external photon lines and internal photon lines.



Feynman Rules (Sec. 7.3)



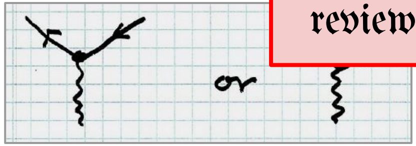
- A Feynman diagram is a contribution to the transition matrix element S_{FI};
- Elements of the diagram are : vertices (=$e\gamma^\mu$) , incoming and outgoing lines (= spinor or polarization vector and exp(± ip.x)), internal lines (= propagators);
- each element has an associated factor;
- Exists an integral d^4x for each vertex.

review

$$\langle F | T \; ie\bar{\psi}\gamma_\mu \psi A^\mu(x) \; ie\bar{\psi}\gamma_\nu \psi A^\nu(y) | I \rangle$$

- **Vertices:**

Associated factor $= i\, e\, \gamma_\mu$



- ***External* electron lines:**

Suppose $|\mathbf{I}\rangle$ has an electron e($\mathbf{p}$,λ).
That must be annihilated by either ψ(x) or ψ(y).

$$\psi(x) = \sum_{\vec{p}} \sum_{\lambda} \left(\frac{2m}{2E\Omega}\right)^{1/2} \left\{ c_\lambda(\vec{p})\, u_\lambda(\vec{p})\, e^{-ip\cdot x} + d_\lambda^\dagger(\vec{p})\, v_\lambda(\vec{p})\, e^{ip\cdot x} \right\}$$

So the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} u_\lambda(\vec{p})\, e^{-ip\cdot x} \quad \text{or} \quad e^{-ip\cdot y}$$

For a positron ***in*** $|F\rangle$ the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} v_\lambda(\vec{p}\,')\, e^{ip'\cdot x} \quad \text{or} \quad e^{ip'\cdot y}$$

Suppose $|\mathbf{F}\rangle$ has an electron e($\mathbf{p}'$,λ′).
That must be created by either $\bar{\psi}$(x) or $\bar{\psi}$(y).

$$\bar{\psi}(x) = \sum_{\vec{p}} \sum_{\lambda} \left(\frac{2m}{2\Omega E}\right)^{1/2} \left[c_\lambda^\dagger(\vec{p})\, \bar{u}_\lambda(\vec{p})\, e^{ip\cdot x} + d_\lambda(\vec{p})\, \bar{v}_\lambda(\vec{p})\, e^{-ip\cdot x} \right]$$

So the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} \bar{u}_{\lambda'}(\vec{p}\,')\, e^{ip'\cdot x} \quad \text{or} \quad e^{ip'\cdot y}$$

For a positron ***in*** $|I\rangle$ the associated factor is

$$\left(\frac{2m}{2\Omega E}\right)^{1/2} \bar{v}_\lambda(\vec{p})\, e^{-ip\cdot x} \quad \text{or} \quad e^{-ip\cdot y}$$

▸ *External photon lines:*

Suppose |I⟩ has a photon γ(**k**,r) ;
r = 1 or 2 only .
That must be annihilated by either $A^\mu(x)$ or $A^\nu(y)$.

$$A^\mu(x) = \sum_{\vec{k}} \sum_{r=0}^{3} \left(\frac{1}{2\omega\Omega}\right)^{1/2} \epsilon_r^\mu(\vec{k}) \left\{ a_r(\vec{k})\, e^{-ik\cdot x} + a_r^\dagger(\vec{k})\, e^{ik\cdot x} \right\}$$

So the associated factor is

$$\frac{1}{\sqrt{2\Omega\omega}}\, \epsilon_r^\mu(\vec{k})\, e^{-ik\cdot x} \quad \text{or} \quad \epsilon^\nu e^{-ik\cdot y}$$

Suppose |F⟩ has a photon γ(**k′**,r′);
r′ = 1 or 2 only .
That must be created by either $A^\mu(x)$ or $A^\nu(y)$.

Then the associated factor is

$$\frac{1}{\sqrt{2\Omega\omega'}}\, \epsilon_{r'}^\mu(\vec{k}')\, e^{ik'\cdot x} \quad \text{or} \quad \epsilon^\nu e^{ik'\cdot y}$$

An incoming line has a factor of exp(-iq.x) and and outgoing line has a factor of exp(+iq.x), where $\hbar q^\mu$ is the 4-momentum.

review

The other fields produce propagators.
(Wick's theorem)

▸ ***Internal electron lines:***

Suppose Wick’s theorem requires the contraction $\psi(x)\ \bar{\psi}(y)$

Then the associated factor is

$$S_F(x - y) =$$

$$= (2\pi)^{-4} \int d^4p\ \ S_F(p)\ \ e^{-ip.(x - y)}$$

▸ ***Internal photon lines:***

Suppose Wick’s theorem requires the contraction $A^{\mu}(x)\ A^{\nu}(y)$

Then the associated factor is

$$D_F^{\mu\nu}(x - y) =$$

$$= (2\pi)^{-4} \int d^4k\ \ D_F^{\mu\nu}(k)\ \ e^{-ik.(x - y)}$$

• ∃ *Integration over x_1 and x_2*

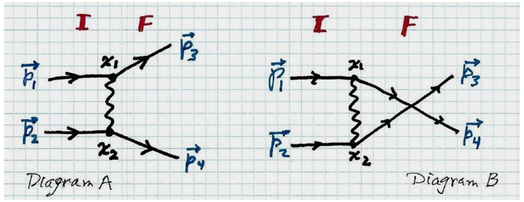


Diagram A

$$\int d^4x_1 \, e^{-i(p_1-p_3)\cdot x_1} e^{-ik\cdot x_1} = \Omega \, \delta_{Kr}(\vec{p}_1+\vec{k};\vec{p}_3) \, 2\pi \, \delta(E_1+k^0-E_3)$$

$$\int d^4x_2 \, e^{-i(p_2-p_4)\cdot x_2} e^{ik\cdot x_2} = \Omega \, \delta_{Kr}(\vec{p}_2-\vec{k};\vec{p}_4) \, 2\pi \, \delta(E_2-k^0-E_4)$$

- 4 momentum is conserved at each vertex
- Evaluate $\int d^4k \Rightarrow k^\mu = p_3^\mu - p_1^\mu = p_2^\mu - p_4^\mu$
- Overall $(2\pi)^4 \, \delta^4(p_3+p_4-p_1-p_2)$

Diagram B is similar (exercise)

Results:

➔ $k^\mu = p_2^\mu - p_4^\mu$ in Diagram A;

➔ $k^\mu = p_2^\mu - p_3^\mu$ in Diagram B;

➔ Overall factor $(2\pi)^4 \, \delta^4(P_F - P_I)$.

The final result, ***in momentum space,*** is

$$S_{FI}^{(2)} = (2\pi)^4 \, \delta^4(p_3+p_4-p_1-p_2) \prod^{(ext)} (m/\Omega E) \, (\mathfrak{M}_a + \mathfrak{M}_b)$$

where

$$\mathfrak{M}_a = -e^2 \, \bar{u}(p_3)\gamma^\mu u(p_1) \, \bar{u}(p_4)\gamma^\nu u(p_2) \; i D_{\mu\nu}(p_2-p_4)$$

and

$$\mathfrak{M}_b = +e^2 \, \bar{u}(p_4)\gamma^\mu u(p_1) \, \bar{u}(p_3)\gamma^\nu u(p_2) \; i D_{\mu\nu}(p_2-p_3)$$

(See Mandl and Shaw, Equation 7.41)

The only question is, why is there a change of sign from $\mathfrak{M}_a$ to $\mathfrak{M}_b$?

$\mathfrak{M}_a$ = the direct term and $\mathfrak{M}_b$ = the exchange term; there is a relative minus sign because of "antisymmetry of the 2-electron wave function".

(But where did it sneak in? Wick's theorem)

Example.

Electron-electron scattering,

$$e(p_1) + e(p_2) \rightarrow e(p_3) + e(p_4)$$

$\Rightarrow$ the Mott cross section;

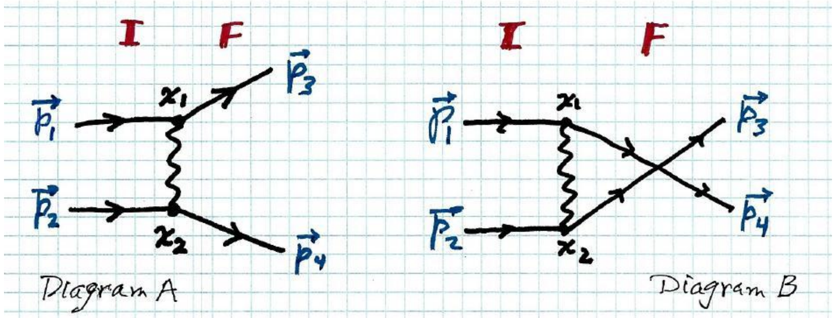
The matrix element

In configuration space,

$$S^{(2)}_{FI} = \langle e3,e4 |\ (-\!- e^2/2!) \iint d^4x_1\, d^4x_2\ T\,[\ (N\psi\gamma^\mu\psi)_{x1}\ (N\psi\gamma^\nu\psi)_{x2}\, A_\mu(x_1)A_\nu(x_2)\ ]\ | e1,e2 \rangle$$

- *Cancellation of 2!*
  - "{e1,e2} are annihilated at {x1,x2}"
    is equal to
  - "{e1,e2} are annihilated at {x2,x1}"
  - Just calculate the case "{e1,e2} are annihilated at {x1,x2}" and multiply by 2!

- *∃ 2 Feynman diagrams (2 vertices)*



- *∃ **a** Photon propagator*

$$\langle 0 |\ T\, A_\mu(x_1)A_\nu(x_2)\ | 0 \rangle = i\, D_{F\mu\nu}\,(x_1 - x_2)$$

$$= i \int d^4k\,/(2\pi)^4\ \exp[-i\, k.(x_1 - x_2)]\ D_{F\mu\nu}(k)$$

- *∃ **4** External electrons*

Diagram A has these factors ...

$$\bar{u}(p_3)\gamma^\mu u(p_1) \quad \times \quad \bar{u}(p_4)\gamma^\nu u(p_2)$$

$$\times \exp[-i\,(p_1 - p_3).x_1\,] \times \exp[-i\,(p_2 - p_4).x_2]$$

(×normalization factors)

Diagram B is similar (exercise)

Appendix B.
Feynman Rules and Formulae
for Perturbation Theory

(i) The Feynman amplitude

The Feynman amplitude ($\mathfrak{M}$) for the transition $|i\rangle \rightarrow |f\rangle$ is defined in terms of the S-matrix element S_{fi} by

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^4 (P_{final} - P_{initial}) \times \prod^{(i)} (2VE_i)^{-½} \prod^{(f)} (2VE_f)^{-½} \prod^{(D)} (2m_{dirac})^{½} \times \mathfrak{M}$$

(Equation 8.1)

(ii) The cross section

The differential cross section for the collision of two particles (i = 1, 2) moving collinearly with relative velocity v_{rel} and resulting in N final particles (f = 1, 2, 3, …, N) is given by

$$d\sigma = (2\pi)^4 \delta^4(P_{final} - P_{initial}) \, [4 E_1 E_2 v_{rel}]^{-1} \prod^{(D)} (2m_{dirac}) \prod^{(f)} d^3p_f /(2\pi)^3 \, | \mathfrak{M} |^2$$

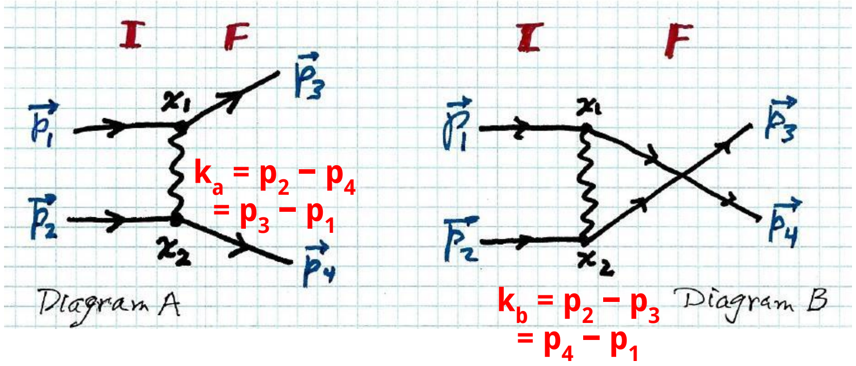
(Equation 8.8)

Example.

Electron-electron scattering,

$$e(p_1) + e(p_2) \rightarrow e(p_3) + e(p_4)$$

$\Rightarrow$ the Mott cross section;



$\mathfrak{M}_a =$

$$= - e^2\, \bar{u}(p_3)\gamma^\mu u(p_1)\, \bar{u}(p_4)\gamma^\nu u(p_2)\, i\, D_{\mu\nu}(k_a)$$

$\mathfrak{M}_b =$

$$= + e^2\, \bar{u}(p_4)\gamma^\mu u(p_1)\, \bar{u}(p_3)\gamma^\nu u(p_2)\, i\, D_{\mu\nu}(k_b)$$

$$D_{\mu\nu}(k) = - g_{\mu\nu} / (k^2 + i\varepsilon);$$

we can set $i\varepsilon = 0$.

To calculate:

$$d\sigma = (2\pi)^4\, \delta^4(P_{final} - P_{initial})$$

$$\times [\, 4\, E_1\, E_2\, v_{rel}]^{-1} \prod^{(D)} (2m_{dirac})$$

$$\times \prod^{(f)} d^3p_f /(2\pi)^3\ |\ \mathfrak{M}_a + \mathfrak{M}_b\ |^2$$

A PHY 955 homework problem:
Mandl and Shaw Problem 7.1