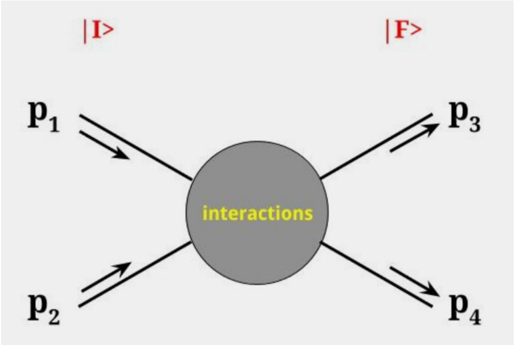
**MANDL AND SHAW**



## Mandelstam variables

Variables s, t, u , defined for a scattering process with 2 incoming particles and 2 outgoing particles;




$$s = ( p_1 + p_2 )^2 = (p_3 + p_4 )^2$$

$$t = ( p_1 - p_3 )^2 = ( p_4 - p_2 )^2$$

$$u = ( p_1 - p_4 )^2 = ( p_3 - p_2 )^2$$

Exercise: Prove

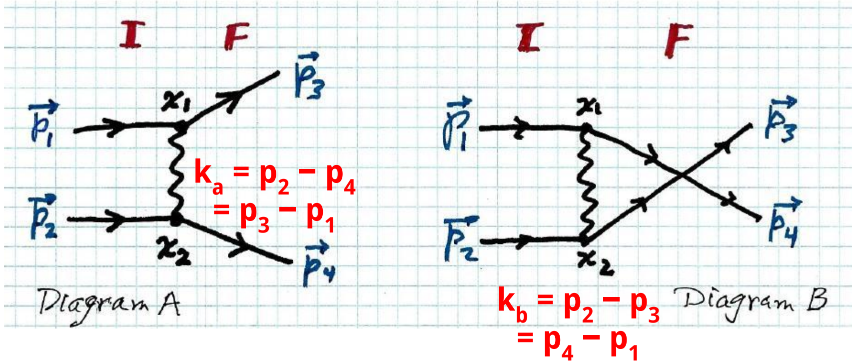
$$s + t + u = m_1^2 + m_2^2 + m_3^3 + m_4^2$$

**Example.**

Electron-electron scattering,

$$e(p_1) + e(p_2) \rightarrow e(p_3) + e(p_4)$$

$\Rightarrow$ the Moller cross section;




$\mathfrak{M}_{\mathfrak{a}}$ =

$$= -\, e^2\, \bar{u}(p_3)\gamma^{\mu}u(p_1)\, \bar{u}(p_4)\gamma^{\nu}u(p_2)\; i\, D_{\mu\nu}(k_a)$$

$\mathfrak{M}_{\mathfrak{b}}$ =

$$= +\, e^2\, \bar{u}(p_4)\gamma^{\mu}u(p_1)\, \bar{u}(p_3)\gamma^{\nu}u(p_2)\; i\, D_{\mu\nu}(k_b)$$

$$D_{\mu\nu}(k) = -\, g_{\mu\nu} \,/\, (\, k^2 + i\varepsilon\, );$$

we can set $i\varepsilon = 0$.

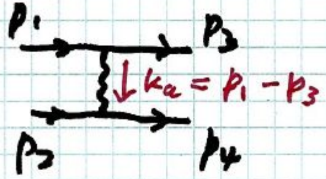
To calculate:

$$d\sigma = (2\pi)^4\, \delta^4(\, P_{\text{final}} - P_{\text{initial}}\, )$$

$$\times [\, 4\, E_1\, E_2\, v_{\text{rel}}]^{-1} \prod^{(D)} (2m_{\text{dirac}})$$

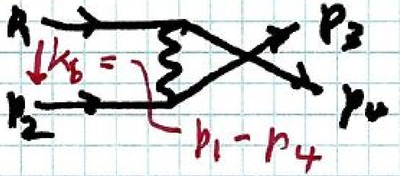
$$\times \prod^{(f)} d^3p_f \,/(2\pi)^3 \;|\; \mathfrak{M}_{\mathfrak{a}} + \mathfrak{M}_{\mathfrak{b}}\; |^2$$

$$\mathcal{M}_a =$$




$$= \frac{e^2}{t}\, \bar{u}_3 \gamma_\mu u_1 \, \bar{u}_4 \gamma^\mu u_2$$

$$\mathcal{M}_b =$$




$$= \frac{-e^2}{u}\, \bar{u}_3 \gamma_\nu u_2 \, \bar{u}_4 \gamma^\nu u_1$$

$$|\mathcal{M}|^2 = |\mathcal{M}_a|^2 + |\mathcal{M}_b|^2 + \mathcal{M}_a \mathcal{M}_b^* + \mathcal{M}_a^* \mathcal{M}_b$$

$$|\mathcal{M}_a|^2 = \frac{e^4}{t^2}\, \bar{u}_3 \underline{\gamma_\mu} u_1 \, \bar{u}_4 \gamma^{\underline{\mu}} u_2 \, \bar{u}_1 \underline{\gamma_\nu} u_3 \, \bar{u}_2 \gamma^{\underline{\nu}} u_4$$

$$= \frac{e^4}{t^2} \left[ \bar{u}_3 \gamma_\mu u_1 \, \bar{u}_1 \gamma_\nu u_3 \right] \left[ \bar{u}_4 \gamma^\mu u_2 \, \bar{u}_2 \gamma^\nu u_4 \right]$$

$$= \frac{e^4}{t^2} \left[ \mathrm{Tr}\, \gamma_\mu \, u_1 \bar{u}_1 \, \gamma_\nu \, u_3 \bar{u}_3 \right] \left[ \mathrm{Tr}\, \gamma^\mu u_2 \bar{u}_2 \gamma^\nu u_4 \bar{u}_4 \right]$$

**The Casimir trick: replace products of matrix elements by traces!**

For unpolarized scattering, we must sum over final polarizations and average over initial polarizations (i.e., spin states); recall, $\sum_{\lambda=1}^{2} u\bar{u} = \frac{\not{p}+m}{2m}$

$$\overline{|\mathcal{M}_a|^2} = \frac{e^4}{4t^2}\left[\mathrm{Tr}\,\gamma_\mu \frac{\not{p}_1+m}{2m}\gamma_\nu \frac{\not{p}_3+m}{2m}\right]\left[\mathrm{Tr}\,\gamma^\mu \frac{\not{p}_2+m}{2m}\gamma^\nu \frac{\not{p}_4+m}{2m}\right]$$

**The Casimir trick !**

$\gamma$ trace identities

$$= \frac{e^4}{4t^2}\frac{4}{(2m)^2}\left[p_{1\mu}p_{3\nu} - g_{\mu\nu}\,p_1\cdot p_3 + p_{3\mu}p_{1\nu} + m^2 g_{\mu\nu}\right]\frac{4}{(2m)^2}\left[p_2^\mu p_4^\nu - g^{\mu\nu}\,p_2\cdot p_4 + p_4^\mu p_2^\nu + m^2 g^{\mu\nu}\right]$$

$$\overline{|\mathcal{M}_a|^2} = \frac{e^4}{4t^2}\frac{16}{(2m)^4}\left[p_1 p_3 - g\, p_1\cdot p_3 + p_3 p_1 + m^2 g\right]_{\mu\nu}$$
$$\left[p_2 p_4 - g\, p_2\cdot p_4 + p_4 p_2 + m^2 g\right]^{\mu\nu}$$

- Lots of tensor algebra; e.g.

$$(p_1 p_3 + p_3 p_1)_{\mu\nu}(p_2 p_4 + p_4 p_2)^{\mu\nu}$$
$$= p_1\cdot p_2\; p_3\cdot p_4 \times 2 + p_1\cdot p_4\; p_3\cdot p_2 \times 2\,;$$

another example: $(m^2 - p_1\cdot p_3)\, g_{\mu\nu}\, (m^2 - p_2\cdot p_4)\, g^{\mu\nu}$
$$= (m^2 - p_1\cdot p_3)(m^2 - p_2\cdot p_4) \times 4$$

- Lots of algebra to reduce this to Mandelstam variables

$$s = (p_1 + p_2)^2 = 2m^2 + 2p_1\cdot p_2 = (p_3 + p_4)^2 = 2m^2 + 2p_3\cdot p_4$$
$$t = (p_1 - p_3)^2 = 2m^2 - 2p_1\cdot p_3 = (p_4 - p_2)^2 = 2m^2 - 2p_4\cdot p_2$$
$$u = (p_1 - p_4)^2 = 2m^2 - 2p_1\cdot p_4 = (p_3 - p_2)^2 = 2m^2 - 2p_3\cdot p_2$$

<u>Use Mathematica to do the algebra.</u>

$$\overline{|\mathcal{M}_a|^2} = \frac{e^4}{4t^2 m^4}\left\{\frac{1}{2}(s-2m^2)^2 + \frac{1}{2}(2m^2-u)^2 + 2m^2 t\right\}$$

**Computer tools:**
**Mathematica and FeynCalc**

$$\overline{|\mathcal{M}_a|^2} = \frac{e^4}{4t^2m^4}\left\{\frac{1}{2}(s-2m^2)^2 + \frac{1}{2}(2m^2-u)^2 + 2m^2 t\right\}$$

Similarly

$$\overline{|\mathcal{M}_b|^2} = \frac{e^4}{4u^2m^4}\left\{\frac{1}{2}(s-2m^2)^2 + \frac{1}{2}(2m^2-t)^2 + 2m^2 u\right\}$$

<u>Comment</u> Because $e_3$ and $e_4$ are identical particles, the transition probability is invariant under the exchange $t \rightleftarrows u$.

# Interference of the two diagrams

$$\mathcal{M}_a = \frac{e^2}{t}\, \bar{u}_3 \gamma_\mu u_1\, \bar{u}_4 \gamma^\mu u_2$$

$$\mathcal{M}_b = -\frac{e^2}{u}\, \bar{u}_3 \gamma_\nu u_2\, \bar{u}_4 \gamma^\nu u_1$$

$$\mathcal{M}_b^* = -\frac{e^2}{u}\, \bar{u}_2 \gamma_\nu u_3\, \bar{u}_1 \gamma^\nu u_4$$

$$\mathcal{M}_a \mathcal{M}_b^* = \frac{-e^4}{tu}\, \bar{u}_3 \gamma_\mu u_1\, \bar{u}_1 \gamma^\nu u_4\, \bar{u}_4 \gamma^\mu u_2 \bar{u}_2 \gamma_\nu u_3$$

$$= \frac{-e^4}{tu} \operatorname{Tr}\left\{ \gamma^\mu u_1 \bar{u}_1 \gamma^\nu u_4 \bar{u}_4\, \gamma_\mu u_2 \bar{u}_2 \gamma_\nu u_3 \bar{u}_3 \right\}$$

$$\overline{\mathcal{M}_a \mathcal{M}_b^*} = \frac{-e^4}{tu} \cdot \frac{1}{4} \cdot \frac{1}{(2m)^4}$$

$$\operatorname{Tr}\left\{ \gamma^\mu (\not{p}_1 + m) \underbrace{\gamma^\nu (\not{p}_4 + m)\, \gamma_\mu\, (\not{p}_2 + m) \gamma_\nu}_{\text{Call this } X_\mu} (\not{p}_3 + m) \right\}$$

Call this $X$.

$$\underbrace{\gamma^\nu (\not{p}_4 + m)\, \gamma_\mu\, (\not{p}_2 + m)\, \gamma_\nu}_{\text{Call this } X_\mu}$$

"Contraction identities" (A, 2)

$$\gamma^\nu \gamma_\nu = 4 \quad ; \quad \gamma^\nu \gamma^\alpha \gamma_\nu = -2\gamma^\alpha \quad ;$$

$$\gamma^\nu \gamma^\alpha \gamma^\beta \gamma_\nu = 4 g^{\alpha\beta} \quad ; \quad \gamma^\nu \gamma^\alpha \gamma^\beta \gamma^\delta \gamma_\nu = -2 \gamma^\delta \gamma^\beta \gamma^\alpha .$$

$$\Rightarrow \quad X_\mu = -2m^2 \gamma_\mu + 4m\,(p_{2\mu} + p_{4\mu}) - 2\not{p}_2 \gamma_\mu \not{p}_4$$

$\mathrm{Tr}\{\gamma^\mu(\not{p}_1+m)\underbrace{\gamma^\nu(\not{p}_4+m)\gamma_\mu(\not{p}_2+m)\gamma_\nu}_{\text{Call this } X_\mu}(\not{p}_3+m)\}$

Call this $Y$.

$$\Rightarrow X_\mu = -2m^2\gamma_\mu + 4m(p_{2\mu}+p_{4\mu}) - 2\not{p}_2\gamma_\mu\not{p}_4$$

$$\Rightarrow Y = \mathrm{Tr}\{\gamma^\mu(\not{p}_1+m)\,X_\mu\,(\not{p}_3+m)\}$$

$$= \text{a whole lot of } \gamma\text{-matrix traces}$$

$$= -8\,p_1\cdot p_2\;4\,p_3\cdot p_4 + 4m^2\;4\,p_2\cdot p_4$$
$$+ 4m^2\;4\,(p_2\cdot p_1 + p_2\cdot p_3 + p_4\cdot p_1 + p_4\cdot p_3)$$
$$+ 4m^2\;4\,p_1\cdot p_3 - 8m^4\cdot 4$$

$\longrightarrow$ *Mandelstam variables*

$$= -8(s-2m^2)(s-6m^2)$$

$$\mathcal{M}_a \mathcal{M}_b^* = \frac{-e^4}{tu}\,\frac{1}{4}\,\frac{1}{(2m)^4}\,(-8)(s-2m^2)(s-6m^2)$$

$$= \frac{e^4}{8\,t\,u\,m^4}\,(s-2m^2)\,(s-6m^2)$$

$$\mathcal{M}_b \mathcal{M}_a^* = \frac{e^4}{8\,t\,u\,m^4}\,(s-2m^2)(s-6m^2) \qquad (t \rightleftarrows u)$$

Easier:
Use FeynCalc

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4m^4}\Big\{\frac{1}{t^2}\Big[\tfrac{1}{2}(s-2m^2)^2 + \tfrac{1}{2}(2m^2-u)^2 + 2m^2 t\Big]$$

$$+ \frac{1}{u^2}\Big[\tfrac{1}{2}(s-2m^2)^2 + \tfrac{1}{2}(2m^2-t)^2 + 2m^2 u\Big]$$

$$+ \frac{1}{tu}(s-2m^2)(s-6m^2)\Big\}$$

**{ G }**

∴ The cross section in the center of mass frame of reference ( $\vec{p}_1 + \vec{p}_2 = 0$ ) is

$$\left(\frac{d\sigma}{d\Omega_3}\right)_{\text{c.o.m.}} = \frac{1}{64\pi^2 s}(2m)^4 \, |\mathcal{M}|^2 \qquad \boxed{\text{EQUATION } 8.18}$$

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↳ in units with $\hbar = 1$ and $c = 1$.

Restore the units

$$\frac{e^2}{4\pi\hbar c} = \alpha = \frac{1}{137} \; ; \quad (\hbar c)^2 = (0.197 \text{ GeV}\cdot\text{fm})^2$$
$$= (1.97)^2 \text{ GeV}^2 \cdot 10^{-4} \text{ barns}$$

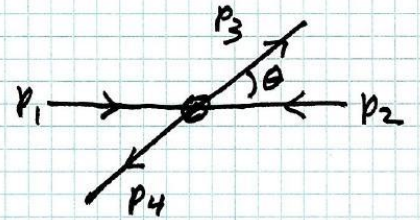
$s, t, u$ have units of $(\text{energy})^2$.

$$\left(\frac{d\sigma}{d\Omega_3}\right)_{C.O.M.} = \frac{(\hbar c)^2}{64\pi^2 s}(2m)^4\,\frac{(4\pi\hbar c\alpha)^2}{4m^4}\left\{\,G\,\right\}\frac{1}{(\hbar c)^2}$$

$$= \alpha^2\,\frac{(\hbar c)^2}{s}\left\{\,G\,\right\}$$

where $s = (p_1 + p_2)^2 = (E_1 + E_2)^2$ in C.O.M. frame

$$= E_{CM}^2$$ s




$$|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_3| = |\vec{p}_4|$$

$$t = 2m^2 - 2p_1\cdot p_3$$

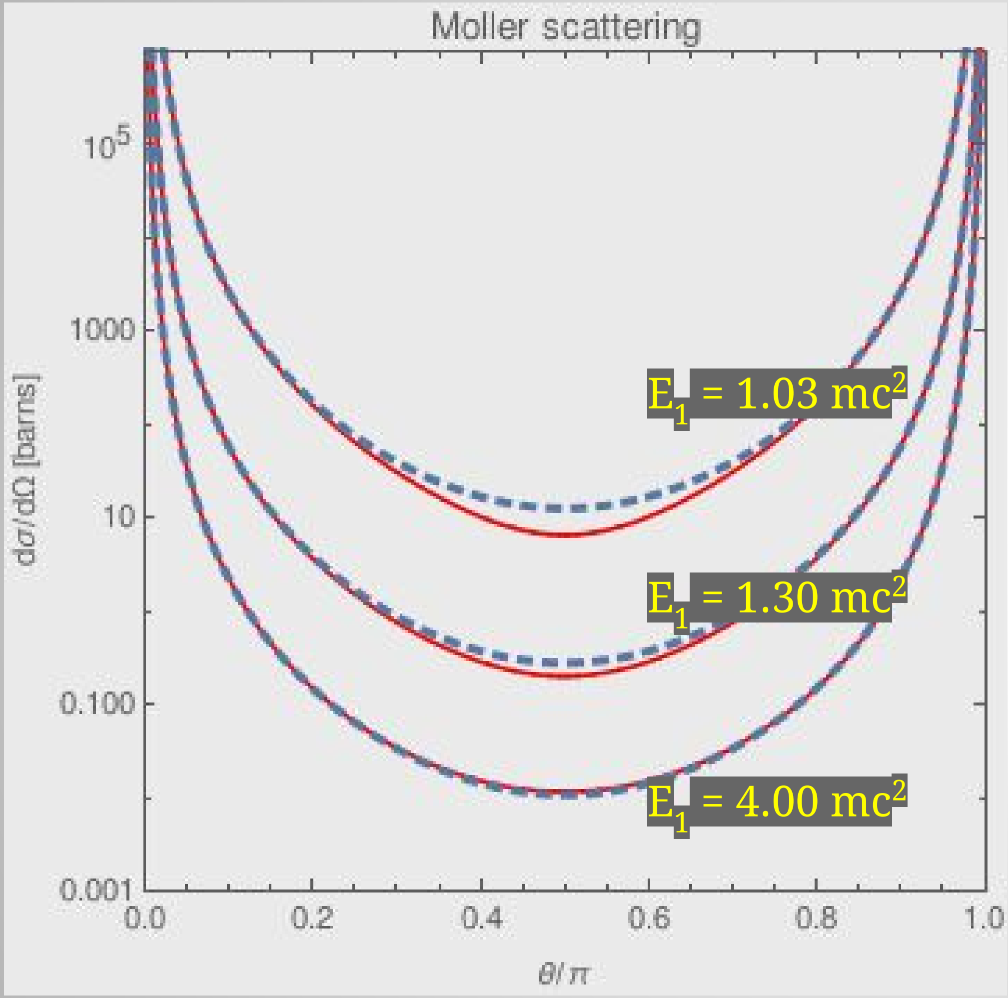
$$= 2m^2 - 2E_1E_3 + 2|\vec{p}_1||\vec{p}_3|\cos\theta$$

$$= 2m^2 - 2E_1^2 + 2|\vec{p}_1|^2\cos\theta$$

$$= -2|\vec{p}_1|^2\{1 - \cos\theta\}$$ t

$$u = 2m^2 - 2p_3\cdot p_2 = -2|\vec{p}_1|^2\{1 + \cos\theta\}$$ u

$$s + t + u = 4E_1^2 - 4|\vec{p}_1|^2 = 4m^2 \checkmark$$




- **Red curves**
  = the Moller cross section;
  CM frame of ref.; E1 = energy of one electron.

- **Blue dashed curves**
  = *a simple approximation*
  $\propto$ $1 / \sin^4 (\theta/2)$ as $\theta \to 0$
  $\propto$ $1 / \sin^4 ((\pi-\theta)/2)$ as $\theta \to \pi$

*based on the classical Rutherford cross section.*

- *Forward - backward symmetry:*
  $\sigma(\pi - \theta) = \sigma(\theta)$
  because $e_3$ and $e_4$ are identical.