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Review and Preview

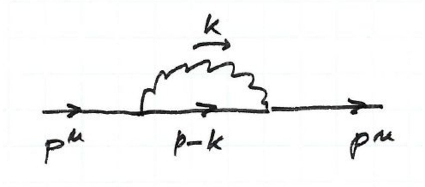
We are testing QED beyond the leading order of perturbation theory. We encounter ...

- IR divergences from soft photons;
- UV divergence in the correction to the photon propagator (vacuum polarization);
- Two more Feynman diagrams will require regularization.

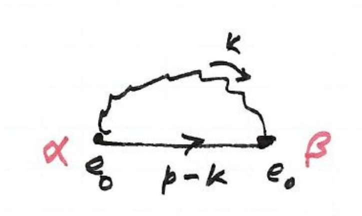
Radiative Corrections, continued

Today it's the electron self-energy insertion.

Wherever an electron propagator appears, the lowest-order correction is



The insertion is



$$i e_0^2 \Sigma^{\alpha\beta}(p, m_0) = (ie_0)^2 \int \frac{d^4k}{(2\pi)^4} \frac{(-ig_{\alpha\beta})}{k^2} \gamma^\alpha \frac{\not{p} - \not{k} + m_0}{(p-k)^2 - m_0^2} \gamma^\beta$$

Useful: $\gamma^\alpha \not{p} \gamma_\alpha = -2\not{p}$ and $\gamma^\alpha \gamma_\alpha = 4.$

The integral is divergent, both IR and UV . To regularize the integral we'll replace

$$\frac{1}{k^2 + i\epsilon}$$

by
$$\frac{1}{k^2 - \lambda^2 + i\epsilon} - \frac{1}{k^2 - \Lambda^2 + i\epsilon}$$

This provides an IR cutoff λ and an UV cutoff Λ.

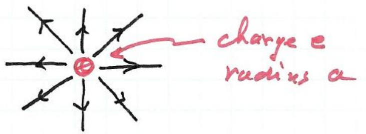
Eventually, $\lambda \to 0$, $\Lambda \to \infty$.

We will encounter mass renormalization ;

$m = m_0 + \delta m$ where $\delta m^{[2]} = e_0^2 K$.

Classical electromagnetic field energy of the electron

In the rest frame,



$\mathbf{E} = \ e/(r^2)\ \mathbf{e}_r$ for $r > a$;

and the field energy is

$$U = \frac{1}{8\pi}\int E^2\, d^3r = \frac{1}{8\pi}\int_a^\infty \frac{e^2}{r^4}\, 4\pi r^2 dr \quad + \text{internal energy}$$

$$= \frac{e^2}{2a} \qquad\qquad \frac{e^2}{4\pi\hbar c} = \alpha = 1/137$$

This has a *linear* divergence in $1/a$ as $a \to 0$.

We'll see that the self-energy in QED is only a logarithmic divergence.

The experimental limit on the electron radius is

$$a < 1.0 \times 10^{-17}\ \text{cm} = 0.0001\ \text{fm}$$

Then the "classical field energy" is

$$U > \frac{4\pi\alpha\,\hbar c}{2a} = \frac{4\pi/137 \times 197\ \text{MeV-fm}}{2 \times 10^{-4}\ \text{fm}}$$

$$\approx 6 \times 10^4\ \text{MeV}$$

And how does that compare to the physical electron mass?

<u>There is a better way to approach mass renormalization.</u>

We have the Lagrangian density

$$\mathfrak{L} = \bar{\psi}(i\gamma.\partial - m_0)\psi + \text{other terms}$$

Rewrite it like this

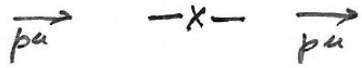
$$\mathfrak{L} = \bar{\psi}(i\gamma.\partial - m)\psi + \bar{\psi}\,(\delta m)\,\psi$$

+other terms

Use this for the unperturbed theory.

Treat this as another "interaction" in the interaction picture.

Now there will be another vertex in the Feynman rules:



$= i\,\delta m$

For example, consider the electron propagator;

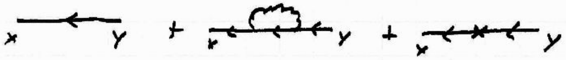
$= \langle \Phi_0 | T\,\psi(x)\,\bar{\psi}(y) | \Phi_0 \rangle$

(Heisenberg picture)

$=$ 

$+ \langle 0 | T\,\psi(x)\,\bar{\psi}(y) \iint e_0(\bar{\psi}\not{A}\psi)\, e_0(\bar{\psi}\not{A}\psi) | 0 \rangle$

$+ \langle 0 | T\,\psi(x)\,\bar{\psi}(y) \int i\delta m\,(\bar{\psi}\psi) | 0 \rangle + h.o.$

$=$ 

in momentum space

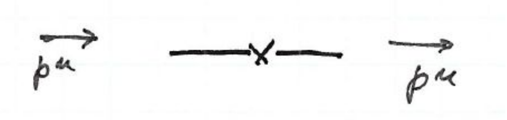
$$= S_0(p) + S_0(p)\; i e_0^2\, \Sigma^{(2)}(p, m)\; S_0(p)$$

$$+\; i\,\delta m\;\; S_0(p)\, S_0(p)$$

The term $L_{\delta m} = -\ \delta m\, \bar{\psi}\psi$
is called the *mass counter term*.

It is required because now we are using the physical mass m in the unperturbed theory instead of the bare mass m_0.

The bare mass disappears from the theory ; but a new interaction appears



"the two-line vertex"

The modified propagator

accurate to order $e_0^{\,2}$

$$\frac{i}{\not p - m + i\epsilon} + \frac{i}{\not p - m + i\epsilon}\, i e_0^2\, \Sigma^{(2)}(p,m)\, \frac{i}{\not p - m + i\epsilon}$$
$$+ \frac{i}{\not p - m + i\epsilon}\, i\,\delta m\, \frac{i}{\not p - m + i\epsilon}$$

$\Sigma(p,m)$ has two spinor indices, and it only depends on p^μ . Therefore it must have the form

$$\Sigma(p,m) = A + (\not p - m)\,[\,B + \Sigma_c(p^2)\,]$$

where A and B are constants and $\Sigma_c(p^2)$ is a scalar with $\Sigma_c(m^2) = 0$.

(Note: $\not p \not p = p^2$)

$$\Sigma^{[2]}(p,m) = A + (\not{p} - m)(B + \Sigma_c)$$

Recall, if $p^2 = m^2$ then $(\not{p}-m)u(p) = 0$.

So,

$$A = \bar{u}(p)\, \Sigma^{[2]}(p,m)\, u(p) \Big|_{p^2 = m^2}$$

(normalization : $\bar{u}u = 1$)

Now there is another geometric series …

$= -\,\delta m/e_0^{\,2}$

to cancel the constant part

$S(p) =$ 

irreducible

$$= \frac{1}{\not p - m}\left[1 + \Sigma \frac{1}{\not p - m} + \Sigma \frac{1}{\not p - m} \Sigma \frac{1}{\not p - m} + \cdots\right]$$

$$= \frac{1}{\not p - m} \frac{1}{1 - \Sigma \frac{1}{\not p - m}} = \frac{1}{\not p - m - (\not p - m)\, \Sigma\, (\not p - m)^{-1}}$$

$$S(p) = \frac{1}{\not p - m} \frac{1}{1 - e_0^2 B - e_0^2 \Sigma_c}$$

$$S(p) = \frac{1}{\not{p}-m}\,\frac{1}{1-e_0^2 B - e_0^2 \Sigma_c}$$

and we need another renormalization

$$= \frac{e_0^2}{(\not{p}-m)(1-e_0^2 B-e_0^2\Sigma_c)}$$

$$= \frac{1}{(\not{p}-m)}\, e_0^2 (1+e_0^2 B)(1+e_0^2 \Sigma_c) + O(e_0^6)$$

should neglect this

$$= \frac{e^2}{\not{p}-m}\left(1+e^2\Sigma_c\right) + O(e^6)$$

where $e^2 = e_0^2(1+e_0^2 B) = e_0^2 Z_2^2$

$$e_0^2 S(p) \sim \frac{e^2}{\not{p}-m} \quad \text{as } p^2 \to m^2$$

is required in order to agree with low-energy scattering, e.g. Thomson scattering.

We write $e^2 = e_0^{\,2} Z_2^{\,2}$.

But is this charge renormalization consistent with $e^{\,2} = e_0^{\,2} Z_3$ *from the photon insertion?*

It *is consistent* to order $e_0^{\,4}$:

$$Z_3^{[2]} (Z_2^{[2]})^2 = (1 + e_0^{\,2}\zeta_{32})(1 + e_0^{\,2}\zeta_{22})^2$$

$$= 1 + e_0^{\,2}(\zeta_{32} + 2\zeta_{22}) + \mathbf{O(e_0^{\,4})}$$

It turns out that there must be ***three*** renormalization constants,

Z_1 , Z_2 , Z_3 ; then $\mathbf{e^2 = e_0^{\,2} Z_3 (Z_2 / Z_1)^2}$.

But $Z_1 = Z_2$, by the Ward identity.

So in the end, $e^2 = e_0^{\,2}\ Z_3$.

More about this next time.

For today, accurate to O(e_0^4),

the bare propagator,

$$\frac{i e_0^2}{\not p - m_0 + i\epsilon}$$

gets replaced by

$$\frac{i e^2}{\not p - m + i\epsilon}\left[1 + e^2 \Sigma_c(p)\right] + O(e^6)$$

<u>Evaluation of δm</u>

On Slide #6, we have

$$\delta m = -\, e_0^2\, A$$

and $A = \bar{u}(p)\, \Sigma(p,m)\, u(p)\, \Big|_{p^2 = m^2}$

$$\therefore\ \delta m = i\, \bar{u}(p) \left\{ -e_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{(-2\not p + 2\not k + 4m)}{(p-k)^2 - m^2 + i\epsilon} \left[\frac{1}{k^2 - \lambda^2 + i\epsilon} - \frac{1}{k^2 - \Lambda^2 + i\epsilon} \right] u(p) \right|_{p^2 = m^2}$$

Also, $\not p\, u(p) = m\, u(p)$ for $p^2 = m^2$

and denominator $= -2p.k + k^2 + i\epsilon$

$$\delta m = i e_0^2\, \bar{u}(p) \int \frac{d^4k}{(2\pi)^4} \frac{2(\not k + m)}{k^2 - 2k.p} \int_{\lambda^2}^{\Lambda^2} \frac{dt}{(k^2 - t)^2}\, u(p)$$

Combine the denominators using a Feynman Integral Formula

$$\frac{1}{a^2 b} = 2\int_0^1 \frac{z\,dz}{[az + b(1-z)]^3}$$

$\Rightarrow$ the combined denominator is

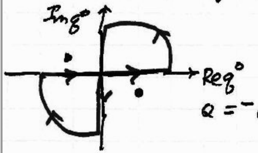
$$(k^2 - t)z + (k^2 - 2k\cdot p)(1-z)$$
$$= k'^2 - \Delta \quad \text{where } \Delta = tz + m^2(1-z)^2$$
$$\text{and } k' = k - p(1-z)$$

Shift the variable of integration from k^μ to $k'^\mu \equiv k^\mu - p^\mu(1-z)$

$$I(k) = \int \frac{d^4k'}{(2\pi)^4} \frac{2(\overset{\to 0}{\not{k}} + \overset{\to m}{\not{p}}(1-z) + m)}{[k'^2 - \Delta + i\epsilon]^3}$$

Now we have a *triple pole* in the complex k^0 plane. Some integral identities,

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - \Delta + i\epsilon)^3} = \frac{i}{(-1)^3}\int \frac{d^4Q}{(2\pi)^4 (Q^2+\Delta)^3}$$



$$= \frac{-i\,2\pi^2}{(2\pi)^4}\int_0^\infty \frac{Q^3\,dQ}{(Q^2+\Delta)^3}$$
$$= \frac{-i}{32\pi^2\Delta}$$

$$\delta m^{(2)} = \frac{m e_0^2}{8\pi^2}\int_0^1 dx \int_{\lambda^2}^{\Lambda^2} dt \,\frac{2z - z^2}{tz + m^2(1-z)^2}$$
$$= \frac{m\alpha_0}{2\pi}\int_0^1 dz\,(2-z)\ln\frac{\Lambda^2 z + m^2(1-z)^2}{\lambda^2 z + m^2(1-z)^2}$$

So…

$$\delta m^{[2]} = \frac{m\alpha_0}{2\pi}\ln\frac{\Lambda^2}{m^2}\int_0^1 dz\,(2-z) + \text{Finite Constant}$$
$$= \frac{3m\alpha_0}{2\pi}\ln\frac{\Lambda}{m} + \text{Finite Constant}$$

$$= \frac{m\alpha_0}{2\pi}\int_0^1 dz\,(2-z)\ln\frac{\Lambda^2 z + m^2(1-z)^2}{\lambda^2 z + m^2(1-z)^2}$$

- Let $\lambda \to 0$, no problem;

mass renormalization is not related to the infrared cutoff (i.e., the fact that $m_\gamma = 0$.)

- Let $\Lambda \to \infty$, get an UV divergence.

$$\delta m^{[2]} \sim \frac{m\alpha_0}{2\pi}\ln\frac{\Lambda^2}{m^2}\int_0^1 dz\,(2-z) + O(1)$$
$$= \frac{3m\alpha_0}{2\pi}\ln\frac{\Lambda}{m} + O(1)$$

The electron mass is $m_0 + \delta m$;
δm is UV divergent, but it's only ***log.*** divergent.

$$\Sigma(p) = A + (\not p - m)B + (\not p - m)\Sigma_c(p^2)$$

$\delta m^{[2]} = -e^2 A$ is log. divergent

<u>What about B?</u>

$Z_2^{[2]} = (1 - e_0^2 B)^{1/2}$

B is ***log.*** divergent like A.

<u>What about $\Sigma_c(p)$?</u>

$\Sigma_c(p)$ is UV convergent.

However, $\Sigma_c(p)$ is *IR divergent*.
But that doesn't bother us, because the Bloch and Nordsieck analysis can handle that.

Summary so far …

The ***photon propagator insertion*** requires charge renormalization by a factor $\mathbf{Z_3}$. The convergent part has physical effects (vacuum polarization). This is an example of "radiative corrections".

The ***electron propagator insertion*** requires the "two-line vertex" (—×—) (i.e., the mass counterterm), and another charge renormalization by a factor $\mathbf{Z_2^{\,2}}$. The convergent part has physical effects; e.g., the dominant contribution to the Lamb shift. This is an example of "radiative corrections".

It seems that we have two charge renormalizations,

$$e^2 = e_0^{\,2}\, Z_3\, Z_2^{\,2}\ .$$

We are only working to order $e_0^{\,4}$ accuracy, so we can replace these "multiplicative" renormalizations by "additive" renormalizations,

$$e^2 = e_0^{\,2}\,(\,1 + e_0^{\,2}\,\Pi^{[2]}\,)\,(\,1 + e_0^{\,2}\,\Sigma^{[2]}\,) + \boldsymbol{O}(e_0^{\,6})$$

$$= e_0^{\,2} + e_0^{\,4}\,\Pi^{[2]} + e_0^{\,4}\,\Sigma^{[2]} + \boldsymbol{O}(e_o^{\,6});$$

the red term cancels the divergent part of the photon-line insertion; the blue term cancels the divergent part of the electron-line insertion.

Finally we need the vertex correction.