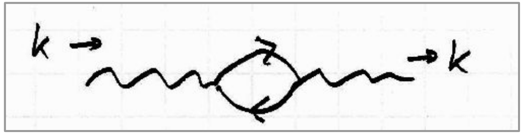
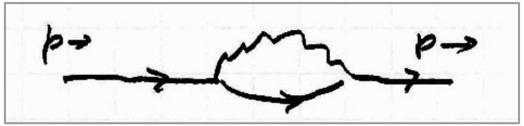
What's next?

In the $O(e_0^{\,2})$ corrections to leading order in QED, there are three Feynman diagrams that require regularization:

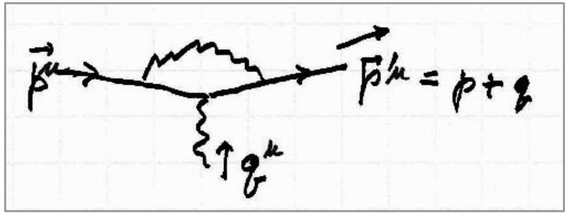
The photon self-energy insertion



The electron self-energy insertion



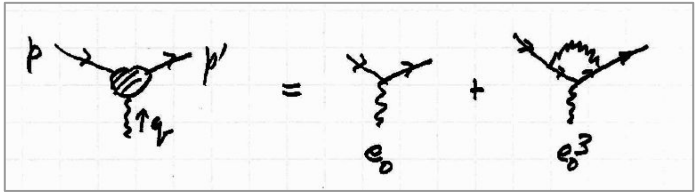
The vertex correction



Radiative Corrections, continued

The Vertex Modification

(Sections 9.5, 9.6.1, and 10.5)

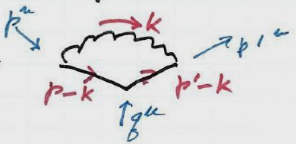


$$ie_0\gamma^\mu \longrightarrow i\Gamma^\mu(p',p)$$

$$\Gamma^\mu(p',p) = e_0\gamma^\mu + e_0^3\Lambda^\mu(p',p) + \cdots$$

(neglect $O(e_0^5)$)

$\Lambda^\mu(p',p) =$



$$= \frac{-i}{(2\pi)^4}\int \frac{d^4k}{k^2+i\epsilon}\,\gamma^\alpha \frac{1}{\not p'-\not k-m}\,\gamma^\mu \frac{1}{\not p-\not k-m}\,\gamma_\alpha$$

physical mass

both UV and IR divergent

∴ replace $\frac{1}{k^2}$ by $\frac{1}{k^2-\lambda^2} - \frac{1}{k^2-\Lambda^2}$

Consider the matrix element

$\bar{u}(p)\,\Lambda^\mu(p,p)\,u(p)$ with $p^2=m^2$

- By Lorentz invariance we can write
  $\bar{u}(p)\Lambda^\mu(p,p)u(p)$
  $= \bar{u}(p)\,(\,a\,\gamma^\mu + b\,p^\mu\,)\,u(p)$

- Use the Gordon decomposition
  $2m\,\bar{u}(p')\,\gamma^\mu\,u(p)$
  $= \bar{u}(p')\,\,[(p'+p)^\mu + i\,\sigma^{\mu\nu}q_\nu]\,\,u(p)$
  where $q^\mu = p'^\mu - p^\mu$.

- Therefore $\bar{u}(p)\Lambda^\mu(p,p)u(p)$
  $= L\,\bar{u}(p)\gamma^\mu u(p)$
  where L is a scalar constant .

- Now write $\Lambda^\mu(p',p) = L\,\gamma^\mu + \Lambda_C{}^\mu(p',p)$.

L is UV divergent; $\Lambda_C{}^\mu$ is UV convergent.

Proof that $\Lambda^\mu{}_C(p',p)$ is convergent:

$$\frac{1}{k^2}\,\frac{1}{\not p' - \not k - m}\,\gamma^\mu\,\frac{1}{\not p - \not k - m}$$

$$\sim \frac{1}{k^4} \quad \text{for large } k^\mu$$

$$\frac{d^4k}{k^4} \sim \frac{k^3\,dk}{k^4} = \frac{dk}{k} \quad \text{is log. divergent}$$

So L is log. divergent.

$$\Lambda^\mu_c(p',p) = \underset{\uparrow \text{log}}{\Lambda^\mu(p',p)} - \underset{\uparrow \text{log}}{L\gamma^\mu}$$

The log divergences cancel, and what is left is UV convergent.

Now we have

$$i\,\Gamma^\mu(p',p)$$

$$= i\,e_0\,[\,\gamma^\mu\,(1 + e_0{}^2\,L) + e_0{}^2\,\Lambda_C{}^\mu(p',p)\,]$$

$$= i\,e_0\,(1 + e_0{}^2\,L\,)\,\gamma^\mu + e_0{}^3\,\Lambda_C{}^\mu(p',p)$$

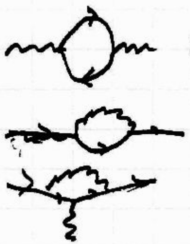
This requires *yet another charge renormalization*, Z_1

$$e = e_0\,(\,1 + e_0{}^2\,L) = e_0\,/Z_1$$

$$i\,\Gamma^\mu(p',p) = i\,e\,\gamma^\mu$$

$$+\,i\,e^3\,\Lambda_c{}^\mu(p',p) + O(e^5)$$

Charge Renormalization



requires $e^2 = e_0^2 Z_3$

requires $e^2 = e_0^2 Z_2^2$

requires $e^2 = e_0^2 / Z_1^2$

If we calculate a physical matrix element to 1-loop order, then all three will occur. That will require

$$e^2 = e_0^2 Z_3 (Z_2/Z_1)^2$$

$$= e_0^2 (1 + e_0^2 \Pi^{[2]}) \left(\frac{1 + e_0^2 \Sigma^{[2]}}{1 + e_0^2 \Lambda^{[2]}} \right)^2 + O(e_0^6)$$

$$= e_0^2 (1 + e_0^2 \Pi^{[2]} + 2 e_0^2 \Sigma^{[2]} - 2 e_0^2 \Lambda^{[2]}) + O(e_0^6)$$

renormalizes photon insertion

renormalizes electron insertion

renormalizes vertex correction

and so the matrix element, written in terms of e^2, will be UV convergent.

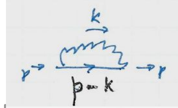
The Ward identity

$$\frac{\partial \Sigma(p)}{\partial p_\mu} = \Lambda^\mu(p,p)$$

e self energy insertion

vertex correction

Now consider



$$\bar{u}(p) \frac{\partial \Sigma(p)}{\partial p_\mu} u(p) = \bar{u}(p) \Lambda^\mu(p,p) u(p)$$

Recall $\Sigma(p) = A + (\not{p} - m) B + (\not{p} - m) \Sigma_c(p^2)$

$\therefore$ LHS $= \bar{u}(p) \{ \gamma^\mu B + \gamma^\mu \Sigma_c + (\not{p} - m) \frac{\partial \Sigma_c}{\partial p^2} 2p^\mu \} u(p)$

$= \bar{u}(p) \gamma^\mu B u(p)$ // $\Sigma_c = 0$ at $p^2 = m^2$; $(\not{p} - m) u(p) = 0$. //

and

RHS $= \bar{u}(p) L \gamma^\mu u(p)$

LHS = RHS $\Rightarrow$ $B = L$

$Z_2^{[2]} = 1 - e_0^2 B = 1 - e_0^2 L = Z_1^{[2]}$ ✓

For 1-loop order, $Z_2^{[2]} = Z_1^{[2]}$; and it is true to all orders; $Z_2 = Z_1$.

$\Lambda_C^{\mu}(p',p)$ and the anomalous magnetic moment of the electron

$\mathfrak{M}$ for electron scattering from a static external field, to lowest order in e, is

$$\mathfrak{M} = i\, e\, \bar{u}(p')\, \gamma_\mu\, u(p)\, A^\mu(q)$$

where $\mathbf{q = p' - p}$

$$= i\, e\, \bar{u}(p')\, [\, p'^\mu + p^\mu + i\, \sigma^{\mu\nu}\, q_\nu \,]\, u(p)\, A^\mu(q)$$

Lorentz force | *magnetic dipole interaction*

To all orders, we can write

$$M = i\, e\, \bar{u}(p') \times [\, \gamma^\mu F_1(q^2) + i\, \sigma^{\mu\nu}\, q_\nu\, F_2(q^2)\, /2m] \times u(p)\ \ A_\mu(q)\, ;$$

electric and magnetic form factors

To 1-loop order,

$$\mathcal{M} = ie\, \bar{u}(p')\left[\gamma^\mu + e^2 \underbrace{\Lambda_c^\mu(p',p)}_{f_1 \gamma^\mu + f_2\, i\sigma^{\mu\nu} q_\nu/2m}\right] u(p)\, A_\mu(q)$$

$$\mathcal{M} = ie\, \bar{u}(p')\left[\gamma^\mu (1 + e^2 f_1) + e^2 f_2\, i\sigma^{\mu\nu} q_\nu/2m\right] u(p)\, A_\mu(q)$$

$$\mathcal{M} \underset{\substack{q^2 \to 0 \\ p^2 \to m^2}}{\sim} ie\, \bar{u}(p')\left[\gamma^\mu + \underbrace{e^2 f_2(q^2=0)}_{\text{the anomalous magnetic moment to 1-loop order of approximation}}\, i\sigma^{\mu\nu} q_\nu/2m\right] u(p)\, A_\mu(q)$$

Calculation of $\Lambda_C^{\mu}(p',p)$

$$\bar{u}(p')\Lambda^{\mu}(p',p)u(p)$$

$$= \frac{-ie_0^2}{(2\pi)^4}\int_{\mathbb{R}} \frac{d^4k}{k^2-\lambda^2}\frac{\gamma^{\alpha}(\not p'-\not k+m)\gamma^{\mu}(\not p-\not k+m)\gamma_{\alpha}}{[(p'-k)^2-m^2][(p-k)^2-m^2]}$$

Combine denominators,

$$\frac{1}{abc} = 2\int_0^1 dy\int_0^{1-y}dz\,\frac{1}{[a+(b-a)y+(c-a)z]^3}$$

Denominator becomes

$$k^2-\lambda^2+[(p'-k)^2-m^2-k^2+\lambda^2]y+[(p-k)^2-m^2-k^2+\lambda^2]z$$

CUBED

$$= k^2-2k\cdot(yp'+zp)-r$$

where $r = \lambda^2(1-y-z)-(p'^2-m^2)y-(p^2-m^2)z$

$$= \lambda^2(1-y-z)$$

$$=(k-a)^2-a^2-r$$

where $a^{\mu} = yp'^{\mu}+zp^{\mu}$

- $d^4k = d^4k'$
- Drop terms liear in k'^{μ}
- Separate $L\gamma^{\mu}$ **from** $\Lambda_C^{\mu}(p',p)$

$$\mathcal{M}_C = e\,\bar{u}(p')\Lambda_C^{\mu}(p',p)u(p)A_{\mu}$$

where

$$\Lambda_C^{\mu} = \frac{-ie^2}{(2\pi)^4}\int_0^1 dy\int_0^{1-y}dz\int\frac{d^4k'\;2N_0^{\mu}}{[(k')^2-a^2-r+i\epsilon]^3}$$

$$N_0^{\mu} = \gamma^{\alpha}(\not p'-\not a+m)\gamma^{\mu}(\not p-\not a+m)\gamma_{\alpha}$$

$$I_{k'} = 2N_0^{\mu}\left(\frac{-i\pi^2}{2}\right)\frac{1}{a^2+r}$$

$$\mathcal{M}_C = \frac{-e^2}{16\pi^2}\int_0^1 dy\int_0^{1-y}dz\,\frac{\bar{u}(p')N_0^{\mu}u(p)}{a^2+r}A_{\mu}$$

Result so far …

$$\mathfrak{M}_{\mathfrak{C}} = \frac{-e^2}{16\pi^2}\int_0^1 dy \int_0^{1-y} dz \; \frac{\bar{u}(p')N_0^{\mu}u(p)}{\alpha^2+r} \; A_{\mu}$$

After a few pages of algebra, and some gamma-matrix tricks, $\mathfrak{M}_{\mathfrak{C}}$ =

$$= \bar{u}(p')\left[\gamma^{\mu} R + i\sigma^{\mu\nu}\frac{q_{\nu}}{2m} S\right] u(p)$$

where R is a scalar, and

where

$$S = \frac{m^2\alpha}{\pi}\int_0^1 dy \int_0^{1-y} dz \; \frac{(y+z)(1-y-z)}{\lambda^2(1-y-z)+(yp'+zp)^2}$$

We can set $\lambda = 0$; *the anomalous magnetic moment does not depend on the IR behavior.*

The limit of low-energy scattering has $q^2 \to 0$; also $p'^{\mu} \to p^{\mu}$ and $p^2 = m^2$;

$$S = \frac{m^2\alpha}{\pi}\int_0^1 dy \int_0^{1-y} dz \; \frac{(y+z)(1-y-z)}{(y+z)^2\, m^2}$$

$$= \frac{\alpha}{\pi}\int_0^1 dy \int_y^1 d\xi \; \frac{1-\xi}{\xi} \qquad (\xi \equiv y+z)$$

$$= \frac{\alpha}{\pi}\int_0^1 dy \,(-\ln y - 1 + y)$$

$$= \frac{\alpha}{\pi}\left(-y\ln y + y - y + y^2/2\right)\Big|_0^1$$

$$= \alpha/2\pi$$

So the matrix element to order e^3 is

$$\mathcal{M} = \frac{ie}{2m}\,\bar{u}(p')\Big[(p'+p)^\mu\,(1+R) + i\sigma^{\mu\nu}q_\nu\,\underbrace{(1+S)}_{\downarrow\ \left(1+\frac{\alpha}{2\pi}\right)}\Big]\,u(p)\,\hat{A}_\mu(q)$$

The magnetic moment of the electron is, to order e^3 ,

$$\mu = \frac{-e}{2m}\left(1+\frac{\alpha}{2\pi}\right)$$

(Schwinger, 1948)

We usually write (units: $\hbar$ = 1 and c = 1)

$$\mu = \frac{-e}{4m}\,g$$

Then the *correction* to the magnetic moment is

$$\delta\mu = \frac{-e}{4m}(g-2)$$

To 1-loop order,

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} + O(\alpha^2)$$

Compare

current theory and current data:

(g – 2)/2 for the electron =

1 159 652 183 ± 8 $\times 10^{-12}$ (thy)

1 159 652 181 ± 7 $\times 10^{-12}$ (exp)

(g – 2)/2 for the muon is also in play.