CHAPTER 10 - TIME EVOLUTION OF QUANTUM SYSTEMS

OUTLINE of the chapter.

Section 10.1 ▸ The Schroedinger representation

You know …

- ∃ a Hilbert space of states (kets)
- Observables are Hermitian operators
- H is the generator of translation in time … …

$$i \frac{\partial}{\partial t} |\psi, t\rangle = H \, |\psi, t\rangle \qquad (\hbar = 1)$$

We need a boundary condition; say $|\psi, 0\rangle = |\psi_0\rangle$.

Solution is $|\psi, t\rangle = e^{-iHt} |\psi_0\rangle$.

Or, for arbitrary time α

$$|\psi, t\rangle = e^{-iHt} e^{iH\alpha} |\psi, \alpha\rangle = U(t, \alpha) \, |\psi, \alpha\rangle$$

where $U(t, \alpha) = e^{-iH(t-\alpha)}$

- $U(t, \alpha)$ is unitary
- $U(t, t_2) \, U(t_2, t_1) = U(t, t_1)$

Section 10.2 ▸
The Heisenberg representation

We define $|\psi\rangle_H = |\psi_0\rangle$, independent of time.

And we require, for any observable B,

$$\langle \psi, t |\, B(t)\, | \psi, t \rangle_H = \langle \psi, t |\, B(t)\, | \psi, t \rangle_S$$

$$\langle \psi_0 |\, B_H(t)\, | \psi_0 \rangle = \langle \psi_0 |\, e^{iHt} B_S e^{-iHt}\, | \psi_0 \rangle$$

so

$$B_H(t) = e^{iHt} B_S e^{-iHt}$$

(In the Schroedinger representation, B does not depend on t.)

Time dependence in terms of diff. eqs.,

$$i \frac{\partial B}{\partial t} = e^{iHt}(-HB_S)e^{-iHt} + e^{iHt}(B_S H)e^{-iHt}$$

$$= -H\,B(t) + B(t)\,H = -[H, B(t)]$$

$$\frac{\partial B}{\partial t} = i\,[H, B(t)]$$

with initial condition $B(0) = B_S$.

For example, consider $H = \frac{p^2}{2m} + V(x)$.

where p is the generator of translation in space

$$f(x+a) = e^{ipa} f(x) e^{-ipa}.$$

$$f(x+\epsilon) = (1 + i\epsilon p) f(x) (1 - i\epsilon p) = f(x) + i\epsilon [p, f]$$

$$\hookrightarrow = f(x) + \epsilon \frac{\partial f}{\partial x} \Rightarrow \frac{\partial f}{\partial x} = i[p, f]$$

i.e., $[p, x] = -i$.

Now, in the Heisenberg picture we have

$$\frac{dp}{dt} = i\left[\frac{p^2}{2m} + V(x),\, p\right]$$

$$= i\,[V(x), p] = -\frac{\partial V}{\partial x}$$

which is Newton's second law for the operators.

Section 10.3 ▸

The interaction representation (or, picture)

Now suppose $H = H_0 + H'$
where H' is a small perturbation of H_0.

We define, for any observable B,

$$B_I(t) = e^{i H_0 t} B_S \, e^{-i H_0 t} \qquad \text{(\#1)}$$

(If H'=0 then I.P. is the same as H.P.)

and we require

$$\langle \psi, t | B(t) | \psi, t \rangle_I = \langle \psi, t | B(t) | \psi, t \rangle_H$$

$${}_I\langle \psi, t | e^{iH_0 t} B_S \, e^{-iH_0 t} | \psi, t \rangle_I$$
$$= \langle \psi_0 | e^{iHt} B_S \, e^{-iHt} | \psi_0 \rangle$$

So $e^{-iH_0 t} | \psi, t \rangle_I = e^{-iHt} | \psi_0 \rangle$

or, $| \psi, t \rangle_I = e^{iH_0 t} | \psi, t \rangle_S \qquad$ (#2)

Or, writing the time evolution in terms of differential equations,

$$i \frac{\partial}{\partial t} | \psi, t \rangle_I = e^{iH_0 t} (-H_0) \, e^{-iHt} | \psi_0 \rangle + e^{iH_0 t} (H) \, e^{-iHt} | \psi_0 \rangle$$
$$= e^{iH_0 t} (H - H_0) \, e^{-iH_0 t} \, e^{iH_0 t} | \psi, t \rangle_S$$
$$= e^{iH_0 t} H' \, e^{-iH_0 t} | \psi, t \rangle_I$$
$$= H'_I(t) | \psi, t \rangle_I \qquad \text{by Equation (1)}$$

Exercise: Calculate $\frac{\partial B}{\partial t}$ in the interaction picture.

Time evolution in perturbation theory

$$i\frac{\partial}{\partial t}|\psi,t\rangle_I = H'_I(t)\,|\psi,t\rangle_I$$

The solution is $|\psi,t\rangle_I = U_I(t,t_0)\,|\psi,t_0\rangle_I$

where

$$U_I(t,t_0) = T\exp\left(-i\right)\int_{t_0}^{t} H'_I(t')\,dt' \qquad \textbf{(\#3)}$$

Proof Assume (3) is true. Then

$$U_I(t+\epsilon,t_0) = U_I(t+\epsilon,t)\,U_I(t,t_0)$$
$$= T\exp(-i)\epsilon H'_I(t)\cdot U_I(t,t_0)$$
$$= [1 - i\epsilon H'_I(t)]\,U_I(t,t_0)$$
$$= U(t,t_0) - i\epsilon H'_I(t)\,U_I(t,t_0)$$

Thus $\frac{\partial U_I}{\partial t} = -i H'_I(t)\,U_I$

$$\frac{\partial}{\partial t}|\psi,t\rangle_I = \frac{\partial U_I}{\partial t}|\psi,t_0\rangle_I$$
$$= -i H'_I(t)\,U_I\,|\psi,t_0\rangle_I$$
$$= -i H'_I(t)\,|\psi,t\rangle_I$$

QED

Result

The time evolution operator, in the interaction picture, is

$$U_I(t,t_0) = T \exp(-i)\int_{t_0}^{t} H_I'(t')\,dt'$$

or

$$U_I(t,t_0) = 1 + \sum_{n=1}^{\infty} (-i)^n \frac{1}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \; T\left[H_I'(t_1)\, H_I'(t_2) \cdots H_I'(t_n) \right]$$

That is the same as Eq. (10.38), which is derived by a different method that emphasizes the time-ordered products.

THE REST OF CHAPTER 10

Section 10.4 – Symmetries and constants of the motion

■ Consider a transformation "R" such that for any states of the system,

$$\langle RB \mid RA \rangle = \langle B \mid A \rangle \ .$$

There is a linear and unitary operator U(R) with

$$\mid RA \rangle = U(R) \mid A \rangle \quad (\textit{any state } \mid A \rangle)$$

and

$$U^{\dagger}(R)\, U(R) = 1.$$

(Time reversal symmetry is different; U(T) is anti-linear and anti-unitary.)

■ Let X be an observable of the system. The transformation of X is

$$X_R = U(R)\, X\, U(R)^{\dagger}$$

because then

$$\langle RB \mid X_R \mid RA \rangle = \langle B \mid X \mid A \rangle \ .$$

■ <u>Theorem 1.</u> If the Hamiltonian H is invariant, i.e., $H_R = H$, then U(R) is a constant of the motion.

$$H_R = U(R)\, H\, U(R)^{\dagger} = H$$

$$U(R)\, H = H\, U(R)$$

$$[\ U(R)\ ,\ H\ ] = 0$$

If H is invariant, then we say that R is a symmetry of the system.

■ <u>Theorem 2.</u> If ∃ a continuous group of symmetry transformations, with generators K_a (a = 1 2 3 …), then the K_a are constants of the motion.

<u>Proof.</u>

The transformation operators are

$$U(\theta_a) = \exp [\ i\ K_a\ \theta_a\ ]$$

and theses are symmetry transformations; so,

$$e^{\,i\,Ka\ \theta a}\ H\ e^{-i\,Ka\ \theta a} = H\ .$$

Thus $[\ K_a\ ,\ H\ ] = 0$.

■ **Example:**

Translation invariance in spacetime.

Let $\Pi(x,t)$ be an observable in the Heisenberg picture.

$$\Pi(x,t) = e^{\,iHt}\ \Pi(x,0)\ e^{-iHt}$$

$$= e^{\,iHt}\ e^{-\,iP.x}\ \Pi(x,0)\ \ e^{+\,iP.x}\ e^{-iHt}$$

In particular consider a matrix element between energy and momentum eigenstates.

$$<\ E_2\ ,\ P_2\ |\ \Pi(x,t)\ |\ E_1\ ,\ P_1\ >$$

$$= \exp\{i(\ (E_2-E_1)t\ -\ (P_2-P_1)x\ )\}<2|\Pi_0|1>$$

What do we learn from this example?

" the matrix element of a local operator between energy and momentum eigenstates has a characteristic spacetime dependence "