# CHAPTER 11 - RELATIVISTIC PERTURBATION THEORY

OUTLINE


Section 11.1 ▸ The Dyson formula

Section 11.2 ▸ Conservation laws

Section 11.3 ▸ Collision cross section and lifetime

Homework ▸


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We want to calculate the rate (⇒ cross section) for collision processes. Consider

$$a + b \rightarrow c + d + e + \ldots$$

$$|\,\text{Initial state}\rangle = |\,a(p_1,\lambda_1)\,;\,b(p_1,\lambda_2)\rangle$$

$$= c_a{}^\dagger\ c_b{}^\dagger\ |\,0\,\rangle$$

$$|\,\text{Final state}\,\rangle = |\,c(p_3,\lambda_3)\,;\,d(p_3,\lambda_3)\,;\,\ldots\ \rangle$$

$S_{FI} = \langle F|\ U_I(t_2\,,\,t_1)\ |\ I\rangle$ in limit $t_1 \rightarrow -\infty\,,\ \ t_2 \rightarrow +\infty$

## Section 11.1 ▸ The Dyson formula

We know

$$U_I(t_2,t_1) = T\exp\,(-i)\int_{t1}^{t2} H'_I(t')\,dt'$$

For example, for QED,

$$£ = £_{free} + £_{int}$$

$$£_{int} = e\,\bar{\psi}\gamma_\mu\psi\,A^\mu$$

Then $H' = -\int £_{int}\,d^3x$

*(Remember, $H = pq - L$.)*

So,

$$S = T\exp i\int £_{int}(x)\,d^4x$$

or,

$$S = \sum i^n/n! \iint\ldots\int d^4x_1\,d^4x_2\ldots d^4x_n$$

$$T\{\,£_{int}(x_1)\,£_{int}(x_1)\ldots £_{int}(x_1)\,\}$$

$$S = 1 + \sum_{n=1} i^n / n! \iint \ldots \int d^4x_1\, d^4x_2 \ldots d^4x_n \; T\{ \mathcal{L}_{int}(x_1)\, \mathcal{L}_{int}(x_2) \ldots \mathcal{L}_{int}(x_n) \} \quad (\bigstar)$$

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*Various comments*

- All the fields that appear in ($\bigstar$) are Interaction Picture operators; i.e., they evolve in time by $H_{free}$ .
- *Lorentz invariance*
  S is a Lorentz invariant operator, because $\mathcal{L}_{int}(x)$ and $d^4x$ are invariant. T (time ordering) is also Lorentz invariant. (*Explain why!*)
- *Normal ordering*
  $$\mathcal{L}_{int} = e : \bar{\psi}\gamma_\mu\psi A^\mu :$$
- Pointlike couplings will eventually require regularization and renormalization.

- The Dyson formula is perfect for perturbation theory.

<u>Second order perturbation theory.</u>

$$S^{(2)} = -\tfrac{1}{2} \iint d^4x_1\, d^4x_2 \; T\{ \mathcal{L}_{int}(x_1)\, \mathcal{L}_{int}(x_2) \}$$

and $\mathcal{L}_{int}(x) = e : \bar{\psi}(x)\gamma_\mu\psi(x)\, A^\mu(x)\;:$

If the initial state has two particles (i.e., for a collision) the final state must have two particles.

Examples with electrons only ...

$$e^- e^- \to e^- e^- \; ; \; e^+ e^+ \to e^+ e^+ \; ;$$
$$e^+ e^- \to e^+ e^- \; ;$$
$$\gamma\, e^- \to \gamma\, e^- \; ; \; e^+ e^- \to \gamma\gamma \; .$$

# Section 11.2 ▸ Conservation laws

- Four momentum is conserved because of a symmetry –– translation invariance in spacetime.

$$I = \int d^4x_1 \dots d^4x_n \, \langle f | \mathcal{L}_{int}(x_1) \dots \mathcal{L}_{int}(x_n) | i \rangle$$

Let $x_i^\mu = X^\mu + r_i^\mu$ where $\sum_{i=1}^{n} r_i^\mu = 0$.

$$I = \int d^4X \; d^4r_1 \dots d^4r_n \; \delta^4(r_1 + \dots + r_n)$$

$$\langle f | \mathcal{L}_{int}(X + r_1) \dots \mathcal{L}_{int}(X + r_n) | i \rangle$$

$$\langle f | e^{iP\cdot X} \mathcal{L}_{int}(r_1) \dots \mathcal{L}_{int}(r_n) e^{-iP\cdot X} | i \rangle$$

$$= e^{i(P_f - P_i)\cdot X} \langle f | \mathcal{L}_{int}(r_1) \dots \mathcal{L}_{int}(r_n) | i \rangle$$

$$I = (2\pi)^4 \delta^4(P_f - P_i) \int d^3r_1 \dots d^3r_n \; \delta^4(r_1 + \dots + r_n)$$

$$\langle f | \mathcal{L}_{int}(r_1) \dots \mathcal{L}_{int}(r_n) | i \rangle$$

∴ $S_{FI}$ will have a factor $(2\pi)^4 \, \delta^4(P_f - P_i)$ .

- Charge Q is conserved because of a symmetry –– gauge invariance.

$$\partial_\mu J^\mu(x) = 0 \quad \Rightarrow \quad Q = \int J^0(x) \, d^3x \text{ is constant.}$$

$$[Q, H] = 0 \quad \text{and} \quad [Q, \mathcal{L}_{int}] = 0.$$

$$0 = \int d^4x_1 \dots d^4x_n \, \langle f | [Q, \mathcal{L}_{int} \dots \mathcal{L}_{int}] | i \rangle$$

$$= (Q_f - Q_i) \int d^4x_1 \dots d^4x_n \, \langle f | \mathcal{L}_{int} \dots \mathcal{L}_{int} | i \rangle$$

∴ $Q_F = Q_I$

## Section 11.3 ▸
## Collision cross section and lifetime

$|I\rangle = |a, b\rangle$; and these are free particles for times long before the collision.

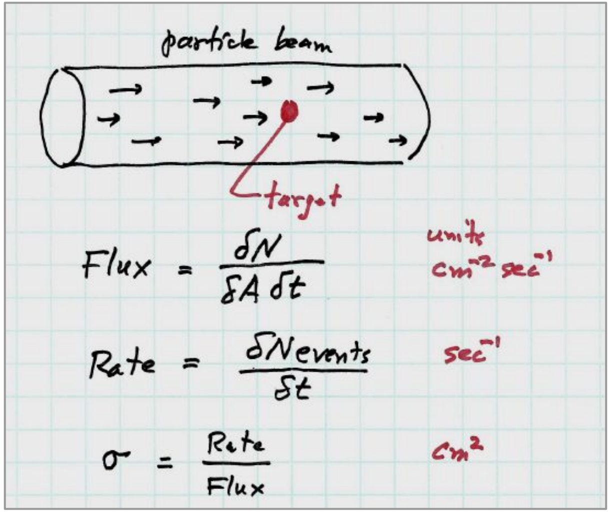
$|F\rangle$ also consists of free particles long after the collision.

To keep the mathematics finite, assume the entire system is enclosed in a finite volume $V = L^3$ ; and apply periodic boundary conditions on the fields.

Also, assume the entire process occurs in a finite time T.

*Eventually, we will take the limits $V \to \infty$ and $T \to \infty$ . All factors of V and T must cancel out in the calculation of the cross section.*

## Flux, Rate and Cross section




*If we assume that every particle that hits the target is an event then $\sigma$ is the cross sectional area of the target. But that is classical thinking.*

**Theorem. (Equation 11.32)**

The *differential cross section* is

$$d\sigma = (NC/v)\; d\Phi_{LIPS}\; |\, \mathcal{M}_{i\to f}\, |^2$$

where $d\Phi$ is the Lorentz-invariant phase space (LIPS) factor

$$d\Phi_{LIPS} = (2\pi)^4\; \delta^4(P_f - P_i) \times \prod^{(out)} d^3p/(2E)/(2\pi)^3\;;$$

and $\mathcal{M}_{i\to f}$ is the Lorentz invariant matrix element defined by

$$\langle f|S|i\rangle = (2\pi)^4\; \delta^4(P_f - P_i) \times \Big[\prod^{(in)} N/\sqrt{V}\Big]\Big[\prod^{(out)} N/\sqrt{V}\Big] \times \mathcal{M}_{i\to f}\;.$$

*Also: **NC** = a normalization constant, and v = relative speed of projectile and target.*

**Proof of the Theorem.**

*( All the notations will be defined in the Proof. )*

- Let $\Phi$ be the *flux* of particles incident on the target. That is, the number of incident particles is $N_{inc} = \Phi \times A_b \times T$ where $A_b$ is the area of the beam.

*(c.g.s. units of $\Phi$ are $cm^{-2}\ sec^{-1}$)*

- Let P be the interaction probability. That is, the number of events $I \to F$ is $N_F = P\, N_{inc}$.

*(Units of P : dimensionless)*

- Then the interaction rate R is $R = N_F\,/T = P\, N_{inc}\,/T$.

*(Units of R are $sec^{-1}$)*

- The interaction cross section is defined by $\Delta\sigma = R / \Phi$.

*(c.g.s. units of $\Delta\sigma$ are $cm^2$)*

Thus

$$\Delta\sigma = \frac{P\, N_{inc}}{T} \times \frac{A_b\, T}{N_{inc}}$$

Also,

$A_b \times (vT)$ = the volume (**area x length**) in which the interactions occurred = V .

So,

$$\Delta\sigma = \frac{P\, V}{v\, T}$$

Quantum Theory: $P = |\langle F|\, S\, |I\rangle|^2$

- *Where are we?* Equation (11.21).

$$\Delta\sigma = \frac{V\, |\langle F|S|I\rangle|^2}{T.v}$$

<u>Conservation of 4-momentum</u>

For infinite volume and infinite time,

$$S_{FI} = (2\pi)^4\, \delta^4(P_f - P_i) \quad \wedge$$

The square of that is undefined.

For finite volume V and time T,

$$(2\pi)^3\, \delta^3(\vec{P}_f - \vec{P}_i) \cong V\, \delta_{Kr}(\vec{P}_f, \vec{P}_i)$$

and (periodic boundary conditions)

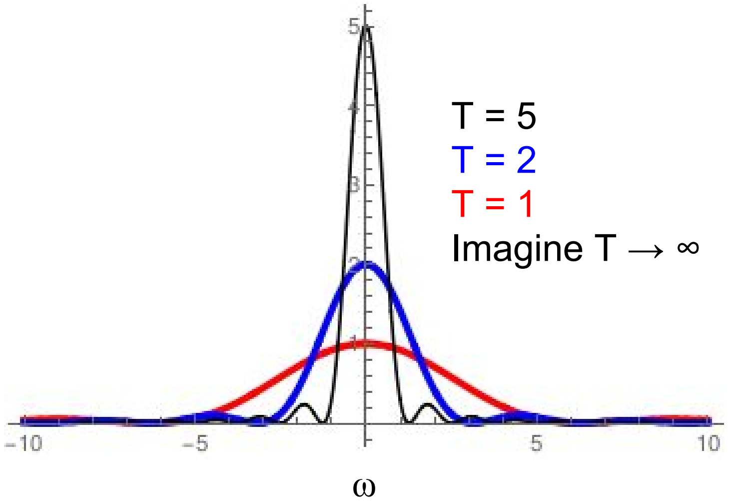
$$(2\pi)\, \delta(E_f - E_i) \cong \int_{-T/2}^{T/2} e^{-i(E_2 - E_1)t}\, dt$$

$$= \frac{2}{\omega} \sin\left(\frac{\omega T}{2}\right) \quad \text{where } \omega = E_1 - E_2 .$$

So we will write

$$|S_{FI}|^2 = V^2\, \delta_{Kr}(\vec{P}_f, \vec{P}_i)\, \frac{4}{\omega^2} \sin^2\left(\frac{\omega T}{2}\right) \quad \wedge\, 2$$

The function $\frac{4}{\omega^2 T} \sin^2\left(\frac{\omega T}{2}\right)$




$\cong 2\pi\,\delta(\omega)$ in limit $T \to \infty$.

So,

$$|S_{FI}|^2 \sim V\,T\,(2\pi)^4\,\delta^4(P_F - P_I)\ |\Lambda|^2$$

The factors of V and T will cancel something else eventually.

Continuing the calculation of the cross section …

We had $\Delta\sigma = \dfrac{V\,|\langle F|S|I\rangle|^2}{T.v}$

so now

$$\Delta\sigma = (1/v)\,V^2\,(2\pi)^4\,\delta^4(P_F - P_I)\ |\Lambda|^2$$

We defined $\Lambda$ like this …

$$S_{FI} = (2\pi)^4\, \delta^4(P_F - P_I)\, \Lambda$$

$S_{FI}$ will also have factors coming from the normalizations of the fields.

For example, consider electron-electron scattering in the tree level approximation (2nd order pert. theory)

$$\langle e_3\, e_4 \mid (-\tfrac{1}{2}) \iint d^4x_1\, d^4x_2 \; T\; \mathcal{L}_{int}(x_1)\; \mathcal{L}_{int}(x_2) \mid e_1\, e_2 \rangle$$

where

$$\mathcal{L}_{int} = e\, \bar{\psi}\, \gamma^\mu\, \psi\, A_\mu$$

Recall the plane wave expansions of $\psi(x)$ and $A^\mu(x)$ ( *known because this is the Interaction Picture*) ; $\Rightarrow$ external factors

| | in $\lvert I\rangle$ | | in $\lvert F\rangle$ |
|---|---|---|---|
| electron | $\sqrt{\frac{m}{EV}}\; u(p)$ | or | $\sqrt{\frac{m}{EV}}\; \bar{u}(p)$ |
| positron | $\sqrt{\frac{m}{EV}}\; \bar{v}(p)$ | or | $\sqrt{\frac{m}{EV}}\; v(p)$ |
| photon | $\frac{1}{\sqrt{2\omega V}}\; \epsilon^\mu(k)$ | or | (same) |

- Collect the *normalization factors* (but not the spinors or polarization vectors!) , and pull them out of $\Lambda$; write

$$\Lambda = \left[\, \prod^{(in)} N/\sqrt{V}\,\right] \left[\, \prod^{(out)} N/\sqrt{V}\,\right] \boldsymbol{M}_{i\to f}\,.$$

- Also, we need to integrate over the final momenta to get the cross section for the specified process

$$\Delta\sigma = \underset{\text{final momenta}}{\left[\sum \ldots \sum\right]} (1/v)\, V^2\, (2\pi)^4\, \delta^4(P_F - P_I)\, |\Lambda|^2$$

- In the limit $V \rightarrow \infty$ ,

  $\sum_{\mathbf{p}}$ becomes $V \int d^3p/(2\pi)^3$

- Putting it all together

$$d\sigma = \frac{1}{V} \left( \prod_{in} \frac{n^2}{2E} \right) (2\pi)^4 \delta^4(P_F - P_I)$$

$$\left( \prod_{F} \frac{d^3p}{(2\pi)^3} \cdot \frac{n^2}{2E} \right) \left| \mathcal{M}_{i \to f} \right|^2$$

where $N = \frac{n}{\sqrt{2E}}$ and $n = \begin{cases} \sqrt{2m} & \text{fermions} \\ 1 & \text{photons} \end{cases}$

*Did you follow what happened to the factors of V? They all cancelled.*

Example $a + b \rightarrow c + d$

$$d\sigma = \frac{n_a^2 n_b^2}{|v_a - v_b| \, 4E_a E_b} (2\pi)^4 \delta^4(P_F - P_I)$$

$$\frac{d^3p_c \, n_c^2}{(2\pi)^3 \, 2E_c} \; \frac{d^3p_d \, n_d^2}{(2\pi)^3 \, 2E_d} \; |\mathcal{M}|^2$$

Equation (11.35).

Homework Problems

due Friday March 24

**2.** Maiani and Benhar, Problem 11.1.

**3.** Maiani and Benhar, Problem 11.2.

**4.** Maiani and Benhar, Problem 11.3.

**5.** Maiani and Benhar, Problem 11.4.