Mandl and Shaw , Section 6.3
WICK'S THEOREM

Motivation

- Wick's theorem is a formal result that will simplify calculations in perturbation theory.
- We need to calculate vacuum expectation values of time ordered products of operators.

Time ordered products

$$T\{ A(x_a)\, B(x_b)\, C(x_c)\, \ldots\, K(x_k) \}$$

$$= \delta_P \; F_1(x_1)\, F_2(x_2)\, F_3(x_3)\, \ldots\, F_{11}(x_{11})$$

where

$$(x_1 , x_2 , x_3 \ldots x_{11}) = P\,(x_a , x_b , x_c \ldots x_k)$$

is the permutation of $(x_a \ldots x_k)$ such that $(x_1 \ldots x_{11})$ are in time order; i.e.,

$$x_1 > x_2 > x_3 > \ldots > x_{11} \,;$$

and

$$(F_1 , F_2 , F_3 \ldots F_{11}) = P\,(A , B , C \ldots K).$$

In words, *the time ordered product is the product of the same operators, with later times to the left of earlier times.* *(δ_P=1 for bosons; δ_P = signature for fermions.)*

Normal ordered products

Write $A = A^{(+)} + A^{(-)}$, $B = B^{(+)} + B^{(-)}$, *etc.*

$$N\{ A^{(sa)} , B^{(sb)} , C^{(sc)} , \ldots , K^{(sk)} \}$$

$$= \delta_P \;\; \mathbf{G_1}^{(+)} \ldots\; \mathbf{G_j}^{(+)} \;\; \mathbf{G_{j+1}}^{(-)} \;\ldots\; \mathbf{G_{11}}^{(-)}$$

In words, *the normal ordered product is the product of the same operators, with creation operators to the left of annihilation ops.* *(δ_P=1 for bosons; δ_P = signature for fermions.)*

Section 6.3.
Wick's theorem

① **LEMMA.** The *time-ordered product* of two fields is equal to the *normal-ordered product* plus a c-number contraction.

$$T\{A(x_1)\,B(x_2)\} =$$

$$N\{A(x_1)\,B(x_2)\} + \underbrace{A(x_1)\,B(x_2)}$$

② The c-number contraction is the propagator.

$$\underbrace{A(x_1)\,B(x_2)} \equiv \langle 0|\ T\{\,A(x_1)\,B(x_2)\,\}\ |0\rangle$$

③ Examples: **these are *all* nonzero contractions**

Real scalar field (6.32a) :

$$\underbrace{\varphi(x_1)\varphi(x_2)} = i\,\Delta_F(\,x_1 - x_2\,)$$

Complex scalar field (6.32b):

$$\underbrace{\varphi(x_1)\varphi^*(x_2)} = \underbrace{\varphi^*(x_2)\,\varphi(x_1)} = i\,\Delta_F(\,x_1 - x_2\,)$$

Dirac field (6.32c):

$$\underbrace{\psi(x_1)\overline{\psi}(x_2)} = -\,\underbrace{\overline{\psi}(x_2)\psi(x_1)} = i\,S_F(\,x_1 - x_2\,)$$

Electromagnetic field (6.32d):

$$\underbrace{A_\mu(x_1)\,A_\nu(x_1)} = i\,D_{F\mu\nu}(\,x_1 - x_2\,)$$

The contraction of two distinguishable fields is 0. *E.g.*, contraction of electron field and quark field = 0.

Homework Problem
due Friday March 31 .

Problem 6.

Use anticommutation relations and definitions to prove, for a Dirac field,

$$T\{\psi_a(\boldsymbol{x})\,\overline{\psi}_b(\boldsymbol{y})\} - N\{\psi_a(\boldsymbol{x})\,\overline{\psi}_b(\boldsymbol{y})\} = i\, S_{F\,ab}\,(\boldsymbol{x}-\boldsymbol{y})$$

④ **THEOREM.** The time-ordered product of *any number of fields* can be written as the sum of all normal ordered products multiplied by c-number contractions. ***(Wick's Theorem)***

⑤ The vacuum expectation value of any normal-ordered product is 0.

⑥ **COROLLARY.** The vacuum expectation value of any time-ordered product is the sum of all complete contractions.

That's where we get Feynman diagrams.

Statement of the theorem

Any time-ordered product of operators can be expressed as the sum of normal-ordered products multiplied by c-number contractions (denoted by χ_{AB}).

$$T\{ A\, B\, C \ldots Z \} = N\{ A\, B\, C \ldots Z \}$$

$$+ \chi_{AB}\, N\{ C\, D\, E \ldots Z \} + \textit{all} \text{ similar terms}$$

$$+ \chi_{AB}\, \chi_{CD}\, N\{ E\, F\, G \ldots Z \} + \textit{all} \text{ similar terms}$$

$$+ \text{all the rest}$$

And why is that useful?

Because the vacuum expectation value of any normal-ordered product is 0.

- $\langle 0 \,|\, N\{ A_1\, A_2 \ldots A_n \} \,|\, 0 \rangle = 0$

because the annihilation terms are ordered to the right, and $\mathbf{A_i}^{(-)} \,|\, 0 \rangle = 0$.

$$\chi_{AB} = \underbrace{A(x)\, B(y)}_{\text{contraction}}$$

Wick's theorem

$$T(U\ V\ W\ \dots\ X\ Y\ Z)$$
$$= N(U\ V\ W\ \dots\ X\ Y\ Z)$$
$$+ \text{all possible contractions.}$$

See quantum field theory textbooks for the general proof. (Mandl/Shaw does not have a general proof; neither does Maiani/Benhar.)

Proof by example (assuming fermions)

Suppose U V W are annihilation parts, at later times than X Y Z, which are creation parts.

Consider

$$\mathbf{Q} = T(U\ V\ W\ \ X\ Y\ Z) = U\ V\ W\ \ X\ Y\ Z.$$

But this is not in normal order.

Move X to the left using the anti-commutation relations; e.g., $WX = -XW + \{W,X\} = -XW + \chi$

$$Q = U\ V\ W\ \ X\ Y\ Z = UV\ (\ \{W,X\} - XW\ )\ YZ$$

$$= -\,UVXWYZ + \chi(W,X)\ UVYZ$$

(the contraction is a c number)

In the first term move X to the left;
in the second term move Y to the left.

$$\mathbf{Q} = -\ (\ -\,UXVWYZ\ \ + \chi(V,X)\ UWYZ\ \ )$$
$$+ \chi(W,X)\ (\ -\,UYVZ + \chi(V,Y)\ UZ\ )$$

Keep going, always moving creation parts to the left;

$$\mathbf{Q} = -\,XUVWYZ + \chi(U,X)\ VWYZ$$
$$-\,\chi(V,X)\ (-UYWZ + \chi(W,Y)\ UZ\ )$$
$$-\,\chi(W,X)\ (-YUVZ + \chi(Y,V)\ UZ\ )$$
$$+ \chi(W,X)\ \chi(V,Y)\ (\ -\,ZU + \chi(U,Z)\ )$$

until all the terms are in normal order.

$$\mathbf{Q} = -\,XYZUVW + \chi(U,X)\ YZVW + \text{many similar}$$
$$-\,\chi(U,X)\ \chi(W,Y)\ ZV + \text{many similar}$$
$$+ \chi(W,X)\ \chi(V,Y)\ (U,Z)\ \ + \text{many similar}$$

= N(UVWXYZ) + all possible ***pairs*** of contractions.

Result

$$T\,(A\,B\,C\,\ldots\,X\,Y\,Z)$$

$$= N\,(A\,B\,C\,\ldots\,X\,Y\,Z)$$

$$+ \sum \chi_{OP}\;\chi_{QR}\;\cdots\;\chi_{ST}\;N\,(A_1\,\ldots\,A_n)$$

These are contractions; c-numbers.

These are left-over operators, in normal order.

Here is the important corollary ...

$$< 0\,|\,T\,(A\,B\,C\,\ldots\,X\,Y\,Z)\,|\,0 >$$

$$= \sum \chi_{OP}\;\chi_{QR}\;\cdots\;\chi_{UV}$$

with no left-over operators.

Now, *what are the contractions?*

☒ Remember that the fields $\psi_a(x)$ and $A_\mu(x)$ { where $x = x^\mu = (t, x, y, z)$ } are operators ***in the interaction picture***.

☒ So we can expand them in plane waves. Consider the electron case (photon case is similar)

$$\psi_i(x) = \sum_{\vec{p},r} \sqrt{\frac{m}{EV}} \Big\{ a_{\vec{p},r}\, u_i(\vec{p},r)\, e^{-ip\cdot x} \quad (\psi^{(-)})$$

$$+\, b^{+}_{\vec{p},r}\, v_i(\vec{p},r)\, e^{ip\cdot x} \Big\} \quad (\psi^{(+)})$$

$i = 1\ 2\ 3\ 4$ = spinor index

$$= \psi_i^{(-)}(x) + \psi_i^{+}(x)$$

☒ Separate the annihilation part and the creation part; $\psi(x) = \psi^{(-)}(x) + \psi^{(+)}(x)$

☒ Now calculate contractions.

$$\chi\left(\psi_i^{(+)}(x),\ \psi_j^{(+)\dagger}(y)\right)$$

$$= T\left[\psi_i^{(+)}(x)\,\psi_j^{(+)\dagger}(y)\right] - N\left[\psi_i^{(+)}(x)\,\psi_j^{(+)\dagger}(y)\right]$$

$$N\left[\psi_i^{(+)}(x)\,\psi_j^{(+)\dagger}(y)\right] = -\,\psi_j^{(+)\dagger}(y)\,\psi_i^{(+)}(x)$$

$$T\left[\psi_i^{(+)}(x)\,\psi_j^{(+)\dagger}(y)\right] = \theta(x^0-y^0)\,\psi_i^{(+)}(x)\,\psi_j^{(+)\dagger}(y) - \theta(y^0-x^0)\,\psi_j^{(+)\dagger}(y)\,\psi_i^{(+)}(x)$$

$$= \theta(x^0-y^0)\left[\psi_i^{(+)}(x)\,\psi_j^{(+)\dagger}(y) + \psi_j^{(+)\dagger}(y)\,\psi_i^{(+)}(x)\right]$$

$$= \theta(x^0-y^0)\left\{\psi_i^{(+)}(x),\ \psi_j^{(+)\dagger}(y)\right\} = \theta(x^0-y^0)\, iS_F(x-y)\,\gamma^0 \quad \text{(HOMEWORK)}$$

Similarly, $\chi\left(\psi^{(-)}(x),\ \psi^{(-)\dagger}(y)\right) = \theta(y^0-x^0)\, iS_F(x-y)\,\gamma^0$

Therefore, $\chi\left(\psi(x),\ \psi^{\dagger}(y)\right) = iS_F(x-y)\,\gamma^0$

$$\chi\left(\psi(x),\ \bar{\psi}(y)\right) = iS_F(x-y) = \langle 0|\,T\,\psi(x)\,\bar{\psi}(y)\,|0\rangle$$

Result:The Wick contraction $\overline{\psi\psi}$ is equal to the propagator function.

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