CHAPTER 14 - APPLICATIONS : QED

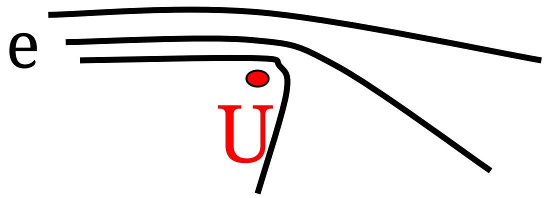
OUTLINE of Chapter 14

Section 14.1 ▸ Scattering in a classical Coulomb field

"We consider the scattering of an electron in a static, i.e., time-independent, external field."

What does that mean?

∃ a static charge or current ;
e.g., a heavy ion—so heavy that we can treat it as a charge fixed at the origin. An electron scatters from the ion.



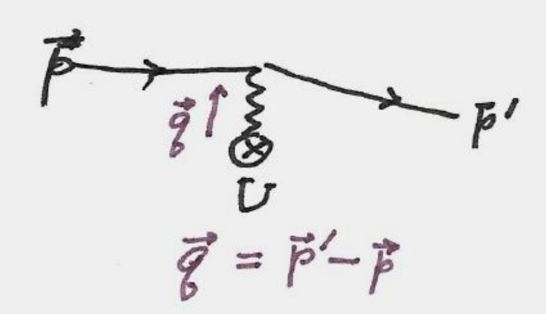
Here is how we start:

$$A^\mu(x) = A^\mu(\vec{x}) = \int \frac{d^3q}{(2\pi)^3}\, \hat{A}^\mu(\vec{q})\, e^{i\vec{q}\cdot\vec{x}}$$

$$S = T\exp\left\{ ie\int d^4x\, \bar{\Psi}(x)\, \not{A}(x)\, \Psi(x) \right\}$$

To the first approximation, we only need the S-matrix element of order 1;

$$S = 1 + ie\int d^4x\, \bar{\Psi}(x)\, \not{A}(x)\, \Psi(x)$$



Energy is conserved, but 3-momentum is not conserved. ***(Why?)***

$$\langle f|S|i\rangle = ie\int d^4x\, \underbrace{\langle p'|\bar{\Psi}(x)\gamma^\mu\Psi(x)|p\rangle}\, A_\mu(x)$$

$$\sqrt{\frac{m}{VE'}}\sqrt{\frac{m}{VE}}\; e^{i(p'-p)\cdot x}\; \bar{u}(p')\gamma^\mu u(p)$$

Note $\int d^4x\; e^{i(p'-p)\cdot x}\; e^{i\vec{q}\cdot\vec{x}}\; \hat{A}^\mu(\vec{q})\, \frac{d^3q}{(2\pi)^3}$

$$= (2\pi)\,\delta(E'-E)\,(2\pi)^3\,\delta^3(\vec{p}+\vec{q}-\vec{p}\,')\,\hat{A}^\mu(\vec{q})\,\frac{d^3q}{(2\pi)^3}$$

$$= (2\pi)\,\delta(E'-E)\,\hat{A}^\mu(\vec{p}\,'-\vec{p})$$

So

$$\langle f|S|i\rangle = 2\pi\,\delta(E'-E)\sqrt{\frac{m}{VE'}}\sqrt{\frac{m}{VE}}\;\mathcal{M}_{fi}$$

where $\mathcal{M}_{fi} = ie\,\bar{u}(\vec{p}\,')\,\not{A}(\vec{p}\,'-\vec{p})\,u(\vec{p})$

Now derive the cross section.

$$d\sigma = \frac{V}{v}\,\frac{|S_{fi}|^2}{T}\sum_{\vec{p}\,'}$$

$$= \frac{V}{vT}\;T\,(2\pi)\,\delta(E'-E)\left(\frac{m}{VE}\right)^2 |\mathcal{M}_{fi}|^2\,\frac{V}{(2\pi)^3}\,d^3p'$$

$\longleftarrow\ |S_{fi}|^2\ \longrightarrow$

$$= \frac{1}{v}\,\frac{m^2}{4\pi^2E^2}\underbrace{\int p'^2\,dp'\,\delta(E'-E)}\,|\mathcal{M}_{fi}|^2\,d\Omega'$$

$$E' = \sqrt{m^2+p'^2}$$

$$dE' = \frac{1}{2}\,\frac{1}{\sqrt{\cdots\cdots}}\,2p'\,dp' = \frac{p'dp'}{E'}$$

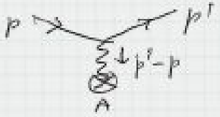
$\therefore$ the integral $= p'E'$ where $\sqrt{p'^2+m^2} = E$

$= pE$ because energy is conserved

$$d\sigma = \frac{1}{v}\,\frac{m^2}{4\pi^2E^2}\,pE\,|\mathcal{M}_{fi}|^2\,d\Omega'$$

$E = \dfrac{m}{\sqrt{1-v^2}}$ and $p = \dfrac{mv}{\sqrt{1-v^2}}$ so $\dfrac{1}{v} = \dfrac{E}{p}$

$$\frac{d\sigma}{d\Omega'} = \frac{E}{p}\,\frac{m^2}{4\pi^2}\,\frac{p}{E}\,|\mathcal{M}_{fi}|^2$$



$$\frac{d\sigma}{d\Omega'} = \frac{m^2}{4\pi^2}\,e^2\,\left|\bar{u}(p')\,\not{A}(p'-p)\,u(p)\right|^2$$

- **The static Coulomb field**

$$A^\mu(\vec{x}) = \frac{Ze}{4\pi|\vec{x}|}\,\delta_{\mu 0}$$

$$\hat{A}^\mu(\vec{q}) = \frac{Ze}{|\vec{q}|^2}\,\delta_{\mu 0}$$

FOR UNPOLARIZED SCATTERING,

$$\frac{1}{2}\sum_{r=1}^{2}\cdot\sum_{r'=1}^{2} \Rightarrow \frac{1}{2}\sum_{r,r'} \bar{u}(p')\gamma^0 u(p) \times \bar{u}(p)\gamma^0 u(p')$$

$$= \frac{1}{2}\,\mathrm{Tr}\,\gamma^0 \sum_{r} u\bar{u}(p)\,\gamma^0 \sum_{r'} u\bar{u}(p')$$

$$= \frac{1}{2(2m)^2}\,\mathrm{Tr}\,\gamma^0(\not{p}+m)\gamma^0(\not{p}'+m)$$

- **Equation 14.14**

$$\left(\frac{d\sigma}{d\Omega'}\right)_{\text{unpol.}} = \frac{m^2}{4\pi^2}\,e^2\left(\frac{Ze}{|\vec{q}|^2}\right)^2\frac{1}{2(2m)^2}\,\mathrm{Tr}\,\gamma^0(\not{p}+m)\gamma^0(\not{p}'+m)$$

$$\frac{d\sigma_{\text{unpol}}}{d\Omega'} = \frac{Z^2e^4}{32\pi^2}\,\frac{1}{|\vec{q}|^4}\,\mathrm{Tr}\,\gamma^0(\not{p}+m)\gamma^0(\not{p}'+m)$$

where $\vec{q} = \vec{p}\,' - \vec{p}$ and $|\vec{p}\,'| = |\vec{p}|$.

Now we need to do some algebra.
It's pretty simple, but let's practice using FeynCalc ...

Mott e + U ⟶ e + U

```
In[67]:= $LoadFeynArts = False;
         << HighEnergyPhysics`FeynCalc`
```

Loading FeynCalc from /home/stump/.Mathematica/Applications/Hig

FeynCalc 8.2.0 For help, type ?FeynCalc, open FeynCalcRef8.nb

```
In[69]:= (* scalar products *)
         ScalarProduct[p1, p1] = m^2;
         ScalarProduct[pf, pf] = m^2;
         ScalarProduct[p1, pf] = En^2 * (1 - v^2 * cos);
```

CROSS SECTION

Start with Eq. (14.14)

```
In[72]:= coeff = α^2 / (qsq^2);
         Asq = (1 / 2) * Tr[(GS[pf] + m).GA[0].(GS[p1] + m).GA[0]];
         Asq = Asq /. {FourVector[p1, 0] → En};
         Asq = Asq /. {FourVector[pf, 0] → En};
         Asq = Expand[Asq]
```

Out[76]= $2\,\mathrm{En}^2\, g^{00}\, v^2 \cos - 2\,\mathrm{En}^2\, g^{00} + 4\,\mathrm{En}^2 + 2\, g^{00}\, m^2$

```
In[77]:= Asq = 2 En^2 + 2 m^2 + 2 En^2 * v^2 * cos
         Asq = Asq /. {m^2 → En^2 * (1 - v^2)};
         Expand[Asq]
```

Out[77]= $2\,\mathrm{En}^2\, v^2 \cos + 2\,\mathrm{En}^2 + 2\, m^2$

Out[79]= $-2\,\mathrm{En}^2\, v^2 + 2\,\mathrm{En}^2\, v^2 \cos + 4\,\mathrm{En}^2$

PLOT THE CROSS SECTION

```
In[80]:= DCS = coeff * Asq;
         DCS = Simplify[DCS /. {qsq → p^2 + p^2 - 2 p^2 * cos}];
         DCS = Simplify[DCS /. {p → En * v, cos → 1 - 2 * sinsq}]
```

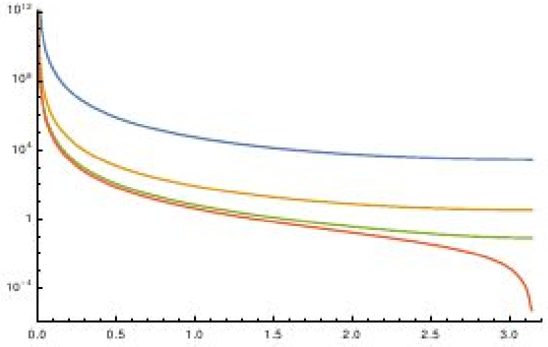
Out[82]= $-\dfrac{\alpha^2\,(\mathrm{sinsq}\, v^2 - 1)}{4\,\mathrm{En}^2\,\mathrm{sinsq}^2\, v^4}$

Mott cross section

$$d\sigma/d\Omega = \frac{\alpha^2\,[\,1 - v^2 \sin^2(\theta/2)\,]}{4\,E_e^{\,2}\, v^4 \sin^4(\theta/2)}$$

```
In[83]:= func = En^2 * DCS;
         func = func /. {α → 1, m → 1, sinsq → Sin[th / 2]^2};
         LogPlot[{
           func /. v → 0.1, func /. v → 0.5,
           func /. v → 0.9, func /. v → 0.99999},
          {th, 0, Pi}, PlotRange → ({0, 3.2}, {1*^-6, 1*^12})]
```

Out[85]=



- **The Mott cross section**

Mott cross section

$$d\sigma/d\Omega = \frac{\alpha^2 [\,1 - v^2 \sin^2(\theta/2)\,]}{4 E_e^{\,2} v^4 \sin^4(\theta/2)}$$

- **Comments**

1) **The nonrelativistic limit**
2) **The ultrarelativistic limit**
3) **The Coulomb divergence**
4) **Nonrelativistic form factor**

$$\frac{\alpha^2}{4 E_e^{\,2} v^4 \sin^4(\theta/2)}$$ *; the Rutherford cross section; same as classical mechanics*

$$\frac{\alpha^2 \cos^2(\theta/2)}{4 E_e^{\,2} \sin^4(\theta/2)}$$ *; vanishes at* $\theta = \pi$

In the limit $\theta \to 0$,

$$\delta\sigma = (d\sigma/d\Omega) \sin\theta \, d\theta \, d\varphi \sim \theta^{-3} \, d\theta \, ;$$

$\mathbf{q} = 0$ *is singular because* $A_0(r)$ *is long range; i.e.,*

$$A^0(r) \propto 1/r \text{ as } r \to \infty$$

An isolated U nucleus is not really feasible.
For scattering from a U atom, the potential would be a **screened Coulomb potential**, *and* $d\sigma/d\Omega$ *would be finite at* $\theta = 0$.

4/ Nonrelativistic form factor

We took $A_0(r) = Ze / (4\pi\, r)$.

This corresponds to a *point* charge.

That is not real, but just an idealization.

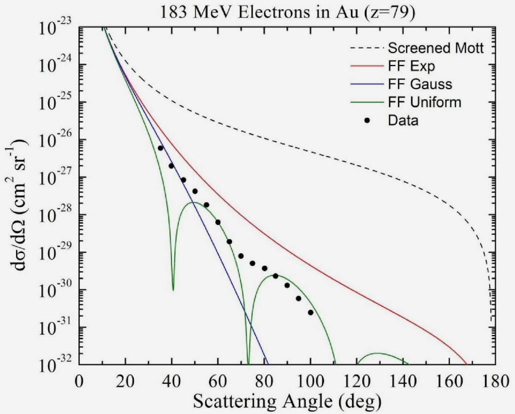
For a charge density $\rho(\mathbf{x}) = Ze\, f(\mathbf{x})$

$$A^0(\mathbf{x}) = \int \rho(\mathbf{y})\, d^3y \,/\, (\, 4\pi\, |\mathbf{x} - \mathbf{y}|)$$

$$\hat{A}_0(\mathbf{q}) = \hat{f}(\mathbf{q})\, Ze \,/\, |\mathbf{q}|^2$$

$$\text{Cross section} = \text{Mott CS} \times |\hat{f}(\mathbf{q})|^2$$

183 MeV Electrons in Au (z=79)



Experimental Data From:
B.Hahn et al. Phys. Rev. 101 (1956) 1131

sr-niel framework:
calculation of the differential cross section without and with three form factors

Homework Problems

due Friday April 7

12. Maiani and Benhar, problem 14.1.1.

13. Mandl and Shaw, problem 8.4.