CHAPTER 14 - APPLICATIONS : QED

OUTLINE of the chapter

Today's lecture is about electron-proton scattering.

The Lagrangian densities for the interactions are

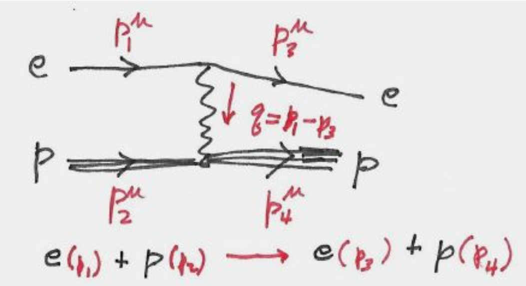
$$\pounds_{int} = e\,\overline{\psi}_e\gamma^\mu\psi_e\ A_\mu(x);$$

and

$$\pounds_{int} = -e\,\overline{\psi}_p[\gamma^\mu + i\kappa\sigma^{\mu\nu}q_\nu/2M]\psi_p\ A_\mu(x),$$

treating the proton as a point particle.

There is one Feynman diagram,



Applying Feynman rules,

$$\boldsymbol{M_{fi}} = e^2\,[\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\Gamma^\nu u(p_2)]\ g_{\mu\nu}/q^2$$

electron *proton*

$$[\,\overline{u}(p_4)\,\Gamma^\mu\,u(p_2)\,]$$

$$\Gamma^\mu = \gamma^\mu + i\,\kappa\,\sigma^{\mu\nu}\,q_\nu\,/2M$$

➪ *Use the Gordon Decomposition* _

$$(\not p - M)\,u(p) = 0 \quad \text{and} \quad \bar u(p')(\not p' - M) = 0$$

$$\bar u(p')\left[\gamma^\mu(\not p - M) + (\not p' - M)\gamma^\mu\right]u(p) = 0$$

$$\bar u(p')\left[\gamma^\mu\gamma^\nu p_\nu + p'^\nu_\nu\,\gamma^\nu\gamma^\mu\right]u(p) = 2M\,\bar u(p')\gamma^\mu u(p)$$

$$\Big[\tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)(p_\nu + p'_\nu)$$

$$+\tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(p_\nu - p'_\nu)\Big]$$

$$= \bar u(p')\left[\,(p^\mu + p'^\mu) + \tfrac{1}{2}(-2i)\,\sigma^{\mu\nu}(-q_\nu)\right]u(p)$$

$q = p' - p$

$$= \bar u(p')\left[(p^\mu + p'^\mu) + i\sigma^{\mu\nu}q_\nu\right]u(p)$$

$$\bar u(p')\,i\sigma^{\mu\nu}q_\nu\,u(p) = \bar u(p')\left\{2M\gamma^\mu - (p^\mu + p'^\mu)\right\}u(p)$$

$$\bar u(p_4)\,\Gamma^\mu\,u(p_2) = \bar u_4\left[\gamma^\mu + \frac{K}{2M}\,i\sigma^{\mu\nu}q_\nu\right]u(p_2)$$

$$= \bar u_4\left[\gamma^\mu(1+K) - \frac{K}{2M}(p_2^\mu + p_4^\mu)\right]u(p_2)$$

or, $$\bar u_4\left[\gamma^\mu(F_1 + F_2) - F_3\,\frac{p_2^\mu + p_4^\mu}{2M}\right]u(p_2)$$

➪ ➪ *Square the matrix element,*

$$\boldsymbol{M_{fi}} = e^2\ [\overline{u}(p_3)\gamma^\mu u(p_1)]\ \ [\overline{u}(p_4)\Gamma^\nu u(p_2)]\ \ g_{\mu\nu}/q^2$$

$$|M|^2 = \frac{e^4}{t^2}\ \ \bar{u}_3 \gamma^\mu u_1\ \ \bar{u}_4 \Gamma_\mu u_2$$

$$\bar{u}_1 \gamma^\alpha u_3\ \ \bar{u}_2 \Gamma_\alpha u_4$$

$$= \frac{e^4}{t^2}\ Tr\left(u_1 \bar{u}_1\ \gamma^\alpha\ u_3 \bar{u}_3\ \gamma^\mu\right)$$

$$Tr\left(u_2 \bar{u}_2\ \Gamma_\alpha\ \ u_4 \bar{u}_4\ \Gamma_\mu\right)$$

➪ ➪ ➪ *Sum and average over spins to obtain the unpolarized cross section,*

$$\sum_r u\bar{u} = \frac{\not{p}+m}{2m}\quad or\quad \frac{\not{p}+M}{2M}$$

$$|M|^2 = \frac{e^4}{t^2}\ \frac{1}{(2m)^2}\ Tr\ (\not{p}_1+m)\ \gamma^\alpha\ (\not{p}_3+m)\ \gamma^\mu$$

$$\frac{1}{(2M)^2}\ Tr\ (\not{p}_2+M)\ \Gamma_\alpha\ (\not{p}_4+M)\ \Gamma_\mu$$

The differential cross section

$$d\sigma = \tfrac{1}{4}\ \ (2\pi)^4\ \ \delta^4(\ p_3 + p_4 - p_1 - p_2\ )$$

$$\times\ m^2\ M^2\ /\ (E_1\ E_2\ E_3\ E_4)$$

$$\times\ d^3p_3\ d^3p_4\ /(2\pi)^6$$

$$e^4\ /(q^2)^2\ \ \times\ \ \boldsymbol{l}^{\ \mu\nu}/(2m)^2 \times \boldsymbol{h}_{\mu\nu}/(2M)^2$$

where the tensors are

$$\boldsymbol{l}^{\ \mu\nu} = Tr\ \{\ (\not{p}_1+m)\ \gamma^\nu\ (\not{p}_3+m)\ \gamma^\mu\ \}$$

$$\boldsymbol{h}_{\mu\nu} = Tr\ \{\ (\not{p}_2+M)\ \Gamma^\nu\ (\not{p}_4+M)\ \Gamma^\mu\ \}$$

where

$$\Gamma^\mu = (1 + \kappa)\ \gamma^\mu - (\kappa/2M)\ (\ p_2^{\ \mu} + p_4^{\ \mu}\ )$$

Form factors (Section 14.2)

So far we have treated the proton as *point-like*.

To take into account the internal structure of the proton, we introduce phenomenological form factors;

either $F_1(q^2)$ and $F_2(q^2)$;

(Dirac and Pauli)

or $G_E(q^2)$ and $G_M(q^2)$.

(Sachs)

How are these defined?

How are the form factors defined?

- For a point-like proton,

$$\bar{u}(p')\left[\gamma^\mu + \frac{K}{2M} i\sigma^{\mu\nu} q_\nu\right] u(p)$$

- Lorentz invariance implies the form

$$\bar{u}(p')\left[F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i}{2M}\sigma^{\mu\nu} q_\nu\right] u(p)$$

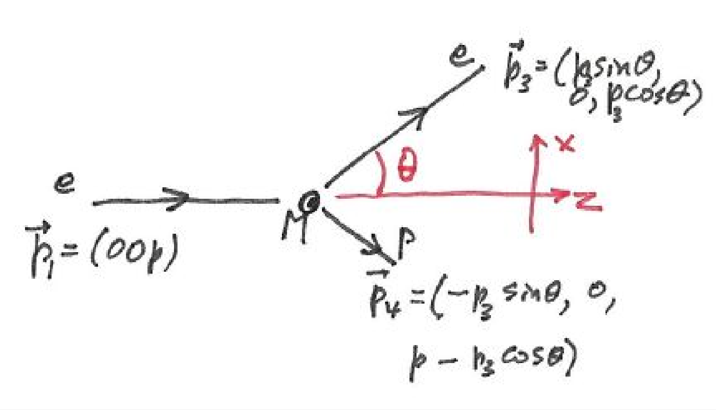
where $F_1(0) = 1$ and $F_2(0) = K$

- Sachs form factors

$$G_E = F_1 + \frac{q^2}{(2M)^2} F_2 \quad ; \quad G_E(0) = 1$$

$$G_M = F_1 + F_2 \quad ; \quad G_M(0) = 1 + K$$

We'll calculate the angular differential cross section , **$d\sigma/d\Omega_3$** , for observation of the final electron , in the *rest frame of the initial proton.*



Kinematics_

$p_1^\mu = (E_1 , 0 , 0 , p)$ where $E_1^2 = p^2+m^2$

$p_2^\mu = (M , 0 , 0 , 0)$

$p_3^\mu = (E_3 , p_3 \sin\theta , 0 , p_3 \cos\theta)$ where $E_3^2 = p_3^2+m^2$

$p_4^\mu = (E_4 , - p_3 \sin\theta , 0 , p - p_3 \cos\theta)$ where $E_4^2 = p_3^2+p^2 - 2pp_3 \cos\theta+M^2$

(Be careful! It can be difficult to distinguish between the various notations; p could mean p^μ or **p** or |**p**|.)

The differential cross section

$$d\sigma = \frac{1}{4}(2\pi)^4 \delta^4(P_F - P_I) \frac{m^2 M^2}{E_1 E_2 E_3 E_4} \frac{d^3p_3\, d^3p_4}{(2\pi)^6} \frac{e^4}{t^2} \frac{l^{\mu\nu} h_{\mu\nu}}{(2m)^2(2M)^2}$$

- The electron mass ($0.511\ MeV/c^2$) is small, so we'll approximate $m = 0$.
- $E_1 = p$ and $E_3 = p_3$ (massless)
- $\int d^3p_4\, \delta^3(P_F - P_I) = 1$ and $\vec{p}_4 = \vec{p}_1 - \vec{p}_3$
- $d^3p_3 = p_3^2\, dp_3\, d\Omega_3$

Conservation of energy $E_4 = p + M - p_3$
and $E_4 = \sqrt{M^2 + p^2 + p_3^2 - 2pp_3\cos\theta}$
implies (exercise)

$$M + p(1 - \cos\theta) = \frac{Mp}{p_3}$$

- $$\int_0^\infty p_3^2\, dp_3\, \delta(p_3 + E_4 - p - M) = \int_0^\infty p_3^2\, dp_3\, \delta[f(p_3)]$$

$$= \frac{p_3^2}{|\partial f/\partial p_3|} \quad ; \quad f(p_3) = p_3 + \sqrt{M^2 + p_3^2 + p^2 - 2pp_3\cos\theta} - p - M$$

$$\frac{df}{dp_3} = 1 + \frac{1}{2}\frac{2p_3 - 2p\cos\theta}{\sqrt{M^2 \cdots \cos\theta}} \leftarrow \text{this is } E_4$$

$$= \frac{1}{E_4}(E_4 + p_3 - p\cos\theta)$$

$$= \frac{1}{E_4}(p + M - p\cos\theta)$$

$$= \frac{p_3^2 E_4}{Mp/p_3} = \frac{p_3^3 E_4}{Mp}$$

Result

$$\frac{d\sigma}{d\Omega_3} = \frac{1}{4} \cdot \frac{1}{4\pi^2} \cdot \frac{1}{16 E_1 E_2 E_3 E_4} \frac{p_3^3 E_4}{Mp} \frac{e^4\, l \cdot h}{t^2}$$

$e^2 = 4\pi\alpha$

$$\frac{d\sigma}{d\Omega_3} = \frac{\alpha^2}{t^2} \frac{p_3^2}{p^2} \frac{l \cdot h}{16 M^2} \quad (14.63)$$

Equation (14.63)

Rosenbluth formula

$$\frac{d\sigma}{d\Omega_3} = \frac{\alpha^2}{t^2} \frac{p_3^2}{p^2} \frac{l \cdot h}{16 M^2} \qquad (14.63)$$

Calculate $l^{\mu\nu} h_{\mu\nu}$ and substitute

$$t = (p_1 - p_3)^2 = m^2 + m^2 - 2E_1E_3 + 2\vec{p}_1 \cdot \vec{p}_3$$
$$= -2pp_3 + 2pp_3 \cos\theta = -2pp_3 \, 2\sin^2\frac{\theta}{2}$$
$$= -4p^2p_3^2 \sin^2\frac{\theta}{2}$$

etc.

Rosenbluth formula for eP → eP

Warning: Only execute this section one time !

```
In[1]:= $LoadFeynArts = False;
<< HighEnergyPhysics`FeynCalc`
```

Loading FeynCalc from /home/stump/.Mathematica/Applications/HighI

FeynCalc 8.2.0 For help, type ?FeynCalc, open FeynCalcRef8.nb

```
In[3]:= (* scalar products *)
ScalarProduct[p1, p1] = m^2;
ScalarProduct[p2, p2] = M^2;
ScalarProduct[p3, p3] = m^2;
ScalarProduct[p4, p4] = M^2;
(* Mandelstam variables *)
ScalarProduct[p1, p2] = (s - m^2 - M^2) / 2;
ScalarProduct[p3, p4] = (s - m^2 - M^2) / 2;
ScalarProduct[p1, p3] = (2 m^2 - t) / 2;
ScalarProduct[p2, p4] = (2 M^2 - t) / 2;
ScalarProduct[p1, p4] = (m^2 + M^2 - u) / 2;
ScalarProduct[p2, p3] = (m^2 + M^2 - u) / 2;
```

|M|^2

```
In[13]:= tensorl = Tr[(GS[p1] + m).GA[α].(GS[p3] + m).GA[μ]];
         tensorh = Tr[
            (GS[p2] + M).((F1 + F2) * GA[α] - F2 / (2 M) * FourVector[p2 + p4, α]).
             (GS[p4] + M).((F1 + F2) * GA[μ] - F2 / (2 M) * FourVector[p2 + p4, μ])];
         ξTT = Contract[tensorl.tensorh]
```

Out[15]=

$$16\left(\frac{1}{2} t\left(F1^2 (2 M^2 - t) + 2 F1^2 t + F1\, F2 (2 M^2 - t) - 2 F1\, F2\, M^2 + 4 F1\, F2\, t + \frac{1}{2} F2^2 (2 M^2 - t) - \frac{F2^2 t (2 M^2 - t)}{8 M^2} - F2^2 M^2 + \frac{7 F2^2 t}{4}\right) + \frac{1}{2} F1^2 (-m^2 - M^2 + s)^2 + \frac{1}{2} F1^2 (m^2 + M^2 - u)^2 + \frac{1}{2} F1^2 t (2 m^2 - t) - F1\, F2 (-m^2 - M^2 + s)(m^2 + M^2 - u) + \frac{1}{2} F1\, F2 (-m^2 - M^2 + s)^2 + \frac{1}{2} F1\, F2 (m^2 + M^2 - u)^2 + F1\, F2\, t (2 m^2 - t) - \frac{F2^2 t (-m^2 - M^2 + s)(m^2 + M^2 - u)}{8 M^2} - \frac{F2^2 t (-m^2 - M^2 + s)^2}{16 M^2} - \frac{1}{2} F2^2 (-m^2 - M^2 + s)(m^2 + M^2 - u) + \frac{1}{4} F2^2 (-m^2 - M^2 + s)^2 - \frac{F2^2 t (m^2 + M^2 - u)^2}{16 M^2} + \frac{1}{4} F2^2 (m^2 + M^2 - u)^2 + \frac{1}{2} F2^2 t (2 m^2 - t)\right)$$

```
In[16]:= ξTT = Expand[ξTT];
         ξTT = Expand[ξTT /. {m^2 → (s + t + u - 2 M^2) / 2}];
         ξTT = Expand[ξTT /. {m^4 → 0}];
         ξTT = Expand[ξTT /. {u → 2 * M^2 - s - t}];
         ξTT = Expand[ξTT /. {s → 2 * M * E1 + M^2}];
         ξTT = Expand[ξTT /. {t → -4 * M^2 * tau}];
         η = Expand[ξTT / M^2] (* Compare Maiani: η = X/M^2 *)
```

Out[22]=

$$64\, E1^2 F1^2 + 64\, E1^2 F2^2 tau - 128\, E1\, F1^2 M\, tau - 128\, E1\, F2^2 M\, tau^2 + 128\, F1^2 M^2 tau^2 - 64\, F1^2 M^2 tau + 256\, F1\, F2\, M^2 tau^2 + 64\, F2^2 M^2 tau^2$$

Rosenbluth formula; Eq. 14.74

```
In[23]:= η1 = Expand[η /. {F1 * F2 → 1 / 2 * (Gm^2 - F1^2 - F2^2)}];
         η1 = Expand[η1 /. {F1^2 → B^2 - tau * F2^2}];
         η1 = Expand[η1 /. {tau^2 → tau * τ, E1 → p}];
         η1 = Expand[η1 /. {tau → p * p3 * sinsq / M^2}];
         η2 = Expand[η1 / (64 * p * p3)];
         η2 = Expand[η2 /. {p / p3 → 1 + 2 p / M * sinsq}];
         ηf = η2 * 64 * E1 * E3
```

Out[29]=

$$64\, E1\, E3 \left(B^2 (-sinsq) + B^2 + 2\, Gm^2\, sinsq\, τ\right)$$

Plot the LAB FRAME cross section; Eq. 14.63; units are MeV and mb

```
In[30]:= dσ = α^2 / (16 t^2) * (E3 / E1)^2 * ηf;
         const = (197)^2 * 1000 / 100;
         (* hbc=197 MeV.fm ; 1 barn = 100 fm^2 *)
         dσ = dσ * const;
```

E1 = incident electron energy in MeV = 400.

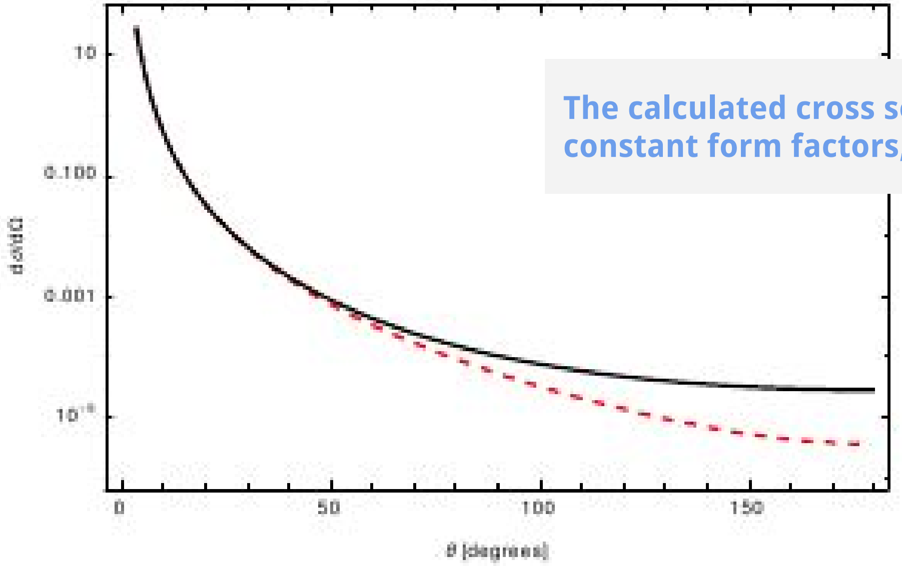
```
In[33]:= Δ = dσ /. {sinsq → (1 / 2) * (1 - cos)};
         Δ = Δ /. {α → 1 / 137, Gm → 1 + κ, B^2 → 1 + τ * κ^2};
         Δ = Expand[Δ /. {τ → -t / (4 M^2)}];
         Δ = Expand[Δ /. {t → -2 * E1 * E3 * (1 - cos)}];
         Δ = Expand[Δ /. {E3 → M * E1 / (M + E1 * (1 - cos))}];
         {Mpr = 938., E1el = 400.};
         Δ = Δ /. {M → Mpr, E1 → E1el};
         Δ = Simplify[Expand[Δ]]
```

Out[39]=

$$\left(0.0000969243\, κ^2 + (0.0000323081\, κ^2 + 0.000129232\, κ - 0.0000869088) \cos^2 + (-0.000129232\, κ^2 - 0.000258465\, κ + 0.000226094) \cos + 0.000129232\, κ + 0.000571467\right) / \left((\cos - 1.)^2 (3.345 - 1. \cos)^2\right)$$

```
In[40]:= Δ0 = Δ /. {κ → 0, cos → Cos[Pi / 180 * angle]};
         Δp = Δ /. {κ → 1.793 , cos → Cos[Pi / 180 * angle]};
         LogPlot[ {Δ0, Δp} , {angle, 0, 180},
          Frame → True, FrameLabel → {"θ [degrees]", "dσ/dΩ"},
          PlotStyle → {{Red, Dashing → {0.01, 0.02}}, Black},
          ImageSize → 256 * 3 / 2]
```

Out[42]=



The calculated cross section, assuming constant form factors, for κ = 0 and κ = 1.793.

Robert Hofstadter was an American physicist. He was the joint winner of the 1961 Nobel Prize in Physics (together with Rudolf Mössbauer) "for his pioneering studies of electron scattering in atomic nuclei and for his consequent discoveries concerning the structure of nucleons". (*Wikipedia*)

Nucleon Electromagnetic Form Factors

2007

C. F. Perdrisat,[1] V. Punjabi,[2] M. Vanderhaeghen [1,3]

Abstract

There has been much activity in the measurement of the elastic electromagnetic proton and neutron form factors in the last decade, and the quality of the data has been greatly improved by performing double polarization experiments, in comparison with previous unpolarized data. Here we review the experimental data base in view of the new results for the proton, and neutron, obtained at MIT-Bates, MAMI, and JLab. The rapid evolution of phenomenological models triggered by these high-precision experiments will be discussed, including the recent progress in the determination of the valence quark generalized parton distributions of the nucleon, as well as the steady rate of improvements made in the lattice QCD calculations.

The form factors decrease with $|q^2|$.

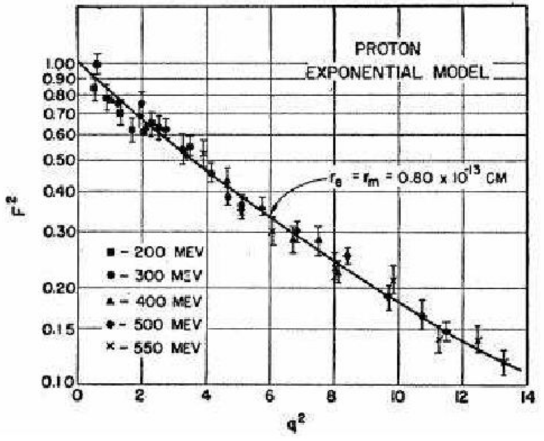


Figure 1: Fig. 27 in [Hof56], with figure caption "The square of the FF plotted against q^2. q^2 is given in units of $10^{26} cm^{-2}$. The solid line is calculated for the exponential model with rms radii=0.80×10^{-13}cm."

Homework not assigned

The next two problems are very difficult.

Use Mathematica to do the algebra.

Mandl and Shaw, problems 8.2 and 8.3.