Calculation of $\vec{E}(\vec{x},t)$ from the Liénard – Wiechert potentials

$$\Phi(\vec{x},t) = \left[\frac{e}{R - \vec{\beta}\cdot\vec{R}}\right]_{ret}$$

$$\vec{A}(\vec{x},t) = \left[\frac{e\vec{\beta}}{R - \vec{\beta}\cdot\vec{R}}\right]_{ret}$$

$\vec{R}(t') = \vec{x} - \vec{r}(t')$ and $\vec{\beta}(t') = \frac{1}{c}\dot{\vec{r}}(t')$

$$t_{ret} = t - \frac{1}{c}\left|\vec{x} - \vec{r}(t_{ret})\right|$$

$$R = \sqrt{R_1^2 + R_2^2 + R_3^2}$$

$$\frac{\partial t_{ret}}{\partial t} = 1 \pm \frac{1}{c}\frac{\vec{R}_0}{R}\frac{\partial}{\partial t}\left(\vec{x} - \vec{r}(t_{ret})\right)$$

($\vec{x}$ fixed)
$$= 1 + \frac{\vec{R}\cdot}{cR}\frac{\partial\vec{r}}{\partial t_{ret}}\frac{\partial t_{ret}}{\partial t}$$

$$= 1 + \frac{1}{c}\hat{R}\cdot\vec{v}\,\frac{\partial t_{ret}}{\partial t}$$

$$\frac{\partial t_{ret}}{\partial t} = \left[\frac{1}{1 - \vec{\beta}\cdot\hat{R}}\right]_{t_{ret}}$$

$$\frac{\partial t_{ret}}{\partial x_i} = \frac{\partial}{\partial x_i}\left\{t - \frac{1}{c}\left|\vec{x} - \vec{r}(t_{ret})\right|\right\}$$

($\vec{t}$ fixed)
$$= -\frac{1}{c}\frac{1}{R}R_j\frac{\partial R_j}{\partial x_i} = -\frac{1}{c}\hat{R}_j\left(\delta_{ij} - v_j\frac{\partial t_{ret}}{\partial x_i}\right)$$

$$= -\frac{1}{c}\hat{R}_i + \vec{\beta}\cdot\hat{R}\,\frac{\partial t_{ret}}{\partial x_i}$$

$$\frac{\partial t_{ret}}{\partial x_i} = \frac{-\hat{R}_i/c}{1 - \vec{\beta}\cdot\hat{R}}$$

We want $\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

$$\Phi = \left[\frac{e}{R - \vec{\beta}\cdot\vec{R}}\right]_{ret}$$

$$-\nabla\Phi = \left[-\nabla\frac{e}{R-\vec{\beta}\cdot\vec{R}}\right]_{ret} + \left[\frac{\vec{R}}{c(R-\vec{\beta}\cdot\vec{R})}\frac{\partial}{\partial t'}\frac{e}{R-\vec{\beta}\cdot\vec{R}}\right]_{ret}$$

$$= e\left[\frac{\hat{R}-\vec{\beta}}{R^2(1-\vec{\beta}\cdot\hat{R})^2}\right]_{ret} + e\left[\frac{\vec{R}}{c(R-\vec{\beta}\cdot\vec{R})^3}\left(c\vec{\beta}\cdot\hat{R} - c\beta^2 + \dot{\vec{\beta}}\cdot\vec{R}\right)\right]$$

$$\left(\vec{R} = \vec{x}-\vec{r}\ ;\ \frac{\partial\vec{R}}{\partial t'} = -\vec{v} = -c\vec{\beta}\ ;\ \frac{\partial R}{\partial t'} = -c\vec{\beta}\cdot\hat{R}\right)$$

$$\vec{A} = \left[\frac{e\vec{\beta}}{R-\vec{\beta}\cdot\vec{R}}\right]_{ret}$$

$$-\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\frac{1}{c}\left[\frac{1}{1-\vec{\beta}\cdot\hat{R}}\frac{\partial}{\partial t'}\frac{e\vec{\beta}}{R-\vec{\beta}\cdot\vec{R}}\right]_{ret}$$

$$= \frac{-e}{c}\left[\frac{\dot{\vec{\beta}}}{R(1-\vec{\beta}\cdot\hat{R})^2} + \frac{\vec{\beta}}{R^2(1-\vec{\beta}\cdot\hat{R})^3}\left(c\vec{\beta}\cdot\hat{R} - c\beta^2 + \dot{\vec{\beta}}\cdot\vec{R}\right)\right]_{ret}$$

after ~ 5 pages of algebra $\Rightarrow$

$$\vec{E} = \underbrace{\frac{e\hat{R}\times\left[(\hat{R}-\vec{\beta})\times\dot{\vec{\beta}}\right]}{cR(1-\vec{\beta}\cdot\hat{R})^3}}_{\substack{\vec{E}_{\text{ACCELERATION}} \\ \sim 1/R}} + \underbrace{\frac{e(\hat{R}-\vec{\beta})(1-\beta^2)}{R^2(1-\vec{\beta}\cdot\hat{R})^3}}_{\substack{\vec{E}_{\text{VELOCITY}} \\ \sim 1/R^2}}$$

$\therefore$ Acceleration of e produces radiation.