1.

Lecture 3
Electromagnetic Plane Waves in Vacuum

We consider fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ in vacuum;

i.e., there are no electric charges nor currents nor materials present,

$\rho(\vec{x},t) = 0$ and $\vec{J}(\vec{x},t) = 0.$

Maxwell's equations are

$$\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = 0$$

.1

Seek solutions in which $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ are both harmonic in time.
Obviously the frequencies must be equal; let the frequency $= \omega/(2\pi)$.

We'll also require $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ to be harmonic functions of x, y, z.
The result is a plane wave.

For example, $\cos(\vec{k} \cdot \vec{x} - \omega t)$.

2.

$$\nabla \cdot \vec{E} = 0 \quad \text{and} \quad \nabla \times \vec{E} = -\tfrac{1}{c}\, \partial \vec{B} / \partial t$$
$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{B} = +\tfrac{1}{c}\, \partial \vec{E} / \partial t$$

Maxwell's equations are real.

Throughout PHY 842 we will do calculations with complex functions.

Just remember...
... in the end we must take the real part of a complex field to be the physical field.

We use complex functions for *mathematical convenience*. It is easier to do calculations with $e^{i\varphi}$ than with $\cos\varphi$ and $\sin\varphi$.

$$e^{i\phi} = \cos\varphi + i \sin\varphi \qquad \text{(Euler)}$$

.2

The Maxwell equations are real, so we can construct complex solutions, with the understanding that the physical solutions are the real part of the complex functions.

Write $\vec{E}(\vec{x}, t) = \vec{E}_0\, e^{i(\vec{k} \cdot \vec{x} - \omega t)}$

and $\vec{B}(\vec{x}, t) = \vec{B}_0\, e^{i(\vec{k} \cdot \vec{x} - \omega t)}$

Note that

$\partial/\partial t\ (\ldots)$ is the same as $-i\omega\ (\ldots)$;

and $\nabla\ (\ldots)$ is the same as $i\vec{k}\ (\ldots)$.

Be sure you understand this!

Result

$$\vec{k} \cdot \vec{E}_0 = 0 \quad \text{and} \quad \vec{k} \times \vec{E}_0 = \omega/c\ \vec{B}_0$$
$$\vec{k} \cdot \vec{B}_0 = 0 \quad \text{and} \quad \vec{k} \times \vec{B}_0 = -\omega/c\ \vec{E}_0$$

3.

$$\vec{k} \bullet \vec{E}_0 = 0 \quad \text{and} \quad \vec{k} \times \vec{E}_0 = \frac{\omega}{c}\, \vec{B}_0$$
$$\vec{k} \bullet \vec{B}_0 = 0 \quad \text{and} \quad \vec{k} \times \vec{B}_0 = -\frac{\omega}{c}\, \vec{E}_0$$

Theorem 1.

$\{ \vec{k}, \vec{E}_0, \vec{B}_0 \}$ form an orthogonal triad of vectors,

with $\vec{E}_0 \times \vec{B}_0$ in the same direction as $\vec{k}$.

Proof.

Start by drawing the vector $\vec{k}$;

then lay down $\vec{E}_0$ perpendicular to $\vec{k}$;

then $\vec{k}$ and $\vec{E}_0$ define a plane ;

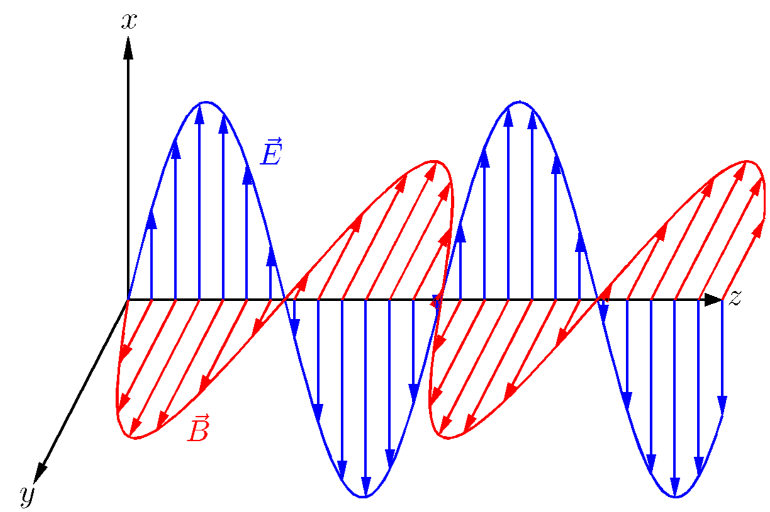
$\vec{B}_0$ is perpendicular to the plane, in the direction of $\vec{k} \times \vec{E}_0$.

Q.E.D.

.3

In[*]:=
```
dir =
  "/Users/OurMacBookAir/Documents/Teaching.2018.current/introduction
   .current/";
Import[StringJoin[dir, "planewave.png"]]
```

Out[*]=



$\{ \vec{k}, \vec{E}_0, \vec{B}_0 \}$ form a right-handed triad of orthogonal vectors.

4.

$$\vec{k} \bullet \vec{E}_0 = 0 \quad \text{and} \quad \vec{k} \times \vec{E}_0 = \frac{\omega}{c} \vec{B}_0$$
$$\vec{k} \bullet \vec{B}_0 = 0 \quad \text{and} \quad \vec{k} \times \vec{B}_0 = -\frac{\omega}{c} \vec{E}_0$$

THE DISPERSION RELATION

= the relation between k ($\equiv |\vec{k}|$) and ω

Theorem 2. $\omega = c\,k$

Proof

$\vec{k} \times (\vec{k} \times \vec{E}_0) = (\omega/c)\,(\vec{k} \times \vec{B}_0)$

l.h.s. $= \vec{k}\,(\vec{k} \,.\, \vec{E}_0) - \vec{E}_0\, k^2 = -k^2\, \vec{E}_0$

and r.h.s. $= -(\omega/c)^2\, \vec{E}_0$

Thus,

$(\omega/c)^2 = k^2$ *Q.E.D.*

The *phase velocity* is $\delta x/\delta t = \omega/k = c$.

.4

This is the starting point for the theory of special relativity.

All electromagnetic waves travel in vacuum with the same speed, c.
(This is both the phase velocity and the group velocity.)

That is, the speed of light (and all e.m. waves) in vacuum does not depend on the observer.

Or, by the principle of relativity, the speed of light in vacuum must be the same in all inertial frames.

From this result, we can derive the transformations between different inertial frames — *the Lorentz transformations.*

5.

Einstein (1905) started with his knowledge of Maxwell's field theory (1866), and from that he deduced the theory of Special Relativity. Interesting history ...

(WT, Chapter 13; Jackson, Chapters 11 and 12)

.5

The ratio B_0 / E_0

Theorem 3.

Calculate B_0 / E_0 for a plane wave in vacuum.

$$\vec{k} \times \vec{E}_0 = (\omega/c)\, \vec{B}_0 \qquad \textit{(F. law)}$$

Say $\vec{B}_0 = B_0\, \hat{b}$; then $\vec{k} \times \vec{E}_0 = \mathrm{k}\, E_0\, \hat{b}$

$$k\, E_0 = (\omega/c)\, B_0$$

$$\Longrightarrow \quad E_0 = B_0$$

Result : $B_0 = E_0$ for a plane wave in vacuum;

★ *equal magnitudes in Gaussian units* ★

6.

What more is there to say about electromagnetic plane waves?

See Homework Set #3.

Problem 3-1

Concerning an electromagnetic plane wave in free space ...

(A) Draw the familiar picture of an electromagnetic wave.

(B) Calculate the energy densities u_E and u_B , and the energy flux density $\vec{S}$. Verify that energy is ***locally*** conserved.

(C) The spectrum of classical waves is infinite; list 7 parts of the EM spectrum.

(D) State the superposition principle.

(E) Explain this statement: The e.m. plane waves (in vacuum) are "complete".

.6

Problem 3-2

Concerning an electromagnetic plane wave in free space ...

(a) Prove $v_{\text{phase}} = \omega/\text{k}$.

(b) Prove $v_{\text{phase}} = \text{c}$.