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Chapter 9. PLANE E. M. WAVES ... AND PROPAGATION IN MATTER

- ★ EM waves in matter
 - ★ Section 9.5
“A simple model for constitutive relations”
 - ★ Continuous media (not truly continuous !)
 - Dielectrics
 - Magnetic materials
 - Metals
 - Plasmas
- Matter is molecular, and molecules contain electric charges and currents.

■ 1

The *microscopic* field equations must include free charge (and free current) and bound charge (and bound current).

For example,

$$\nabla \cdot \vec{E} = 4\pi (\rho_{\text{free}} + \rho_{\text{bound}})$$

where $\rho_{\text{bound}} = -\nabla \cdot \vec{P}$;

$\vec{P}(\vec{x}, t)$ = polarization = dipole moment per unit volume = $\vec{p}(\vec{x}, t) n(\vec{x}, t)$.

We replace ρ_{bound} and \vec{J}_{bound} by introducing \vec{D} and \vec{H} . But then we need constitutive relations. For static fields, $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$ where ϵ and μ are real constants. That should also work for “low frequencies” (*meaning what?*)

For time-dependent fields, it is more complex.

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THE MODEL FOR BOUND CHARGE (AND BOUND CURRENT) IN A SINGLE MOLECULE

Schwinger calls this the “Fermi model”. It’s simplified, but it gives useful results.

■ We'll say a molecule consists of positive charge at rest and negative charge moving according to an equation of motion,

$$m [\ddot{\vec{x}} + \gamma \dot{\vec{x}} + \omega_0^2 \vec{x}] = -e \vec{E}(t)$$

($-e = \text{“effective negative charge”}$)
($m = \text{“effective mass of neg. charge”}$)

■ How can such a simple model work?

It contains the right physics. It will describe the mean dipole moment, averaged over a small volume containing a large number of molecules, sufficiently well. I.e., it gives a reasonable model for $\vec{P}(\vec{x}, t)$.

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■ We are dealing with a tricky limit:
molecular size \ll microscopic length \ll macroscopic length

■ A phenomenological model with **four adjustable parameters**: $m = \text{“mass”}$, or inertia ; $-e = \text{effective negative charge}$; $\gamma = \text{a damping factor}$; $\omega_0/(2\pi) = \text{a frequency for oscillations around equilibrium}$.

■ Note that \vec{x} represents the displacement from equilibrium; i.e.,
 $\vec{x} = \langle \vec{x}_{\text{neg}} - \vec{x}_{\text{pos}} \rangle$ (points from pos to neg); therefore *the dipole moment of the model molecule* is $\vec{p} = -e \vec{x}(t)$.

■ There is a resistive force = $-m \gamma \dot{\vec{x}} \iff$ damping .

■ There is a restoring force = $-m \omega_0^2 \vec{x} \iff$ oscillations around equilibrium

3✓

$$m [\ddot{\vec{x}} + \gamma \dot{\vec{x}} + \omega_0^2 \vec{x}] = -e \vec{E}(t)$$

THE SOLUTION

We are very familiar with this equation. *It is the damped driven oscillator.* The general solution is

$\vec{x}(t)$ = any particular solution
+ the general solution of the homog. eq.

Suppose $\vec{E}(t) = \mathcal{E}(t) \hat{e}_x$.

Then we have a 1D equation,

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = -(e/m) \mathcal{E}(t)$$

The solution is

$$x(t) = \text{a particular solution} + c_1 e^{p_1 t} + c_2 e^{p_2 t}$$

■ The homogeneous solution:

$$e^{pt} \text{ where } p^2 + \gamma p + \omega_0^2 = 0;$$

There are two solutions p_1 and p_2 ;

$$c_1 e^{p_1 t} + c_2 e^{p_2 t} = \text{the transient}.$$

■3

The transient is needed to match whatever initial conditions are given. We are not very interested in the transient.

■ Now we need a particular solution. It will depend on $\mathcal{E}(t)$. For harmonic fields, we can use the *steady-state solution*.

So, now we'll specialize to a harmonic electric field.

$$\text{Write } \mathcal{E}(t) = \mathcal{E}(\omega) e^{-i\omega t}$$

(Real part is implied!)

There is a tricky point here: $\vec{E}(t)$ does not depend on the displacement from equilibrium, \vec{x} . We are assuming that the molecule is much smaller than the wavelength of the e.m. waves. Does that make sense as an approximation? Yes; because for example, $0.1 \text{ nm} \ll 500 \text{ nm}$.

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = -(e/m) \mathcal{E}(\omega) e^{-i\omega t}$$

The steady state solution is $x(t) = R e^{-i\omega t}$.

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(The steady-state oscillations have the same frequency as the driving force.)

Plug $x(t) = R e^{-i\omega t}$ into the equation of motion,

$$[-\omega^2 - i\gamma\omega + \omega_0^2] R = -(e/m) \mathcal{E}(\omega)$$

$m(-)=$

$$\text{So, } R(\omega) = \frac{-(e/m) \mathcal{E}(\omega)}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

Model $x(t) = \frac{-(e/m) \mathcal{E}(\omega)}{\omega_0^2 - \omega^2 - i\gamma\omega} e^{-i\omega t}$

(Re implied)

■4

■ AN IDEAL DIELECTRIC; I.E., A NONCONDUCTING MATERIAL

For low frequencies, i.e., $\omega \ll \omega_0$ and $\omega \ll \gamma$, we can say that the polarization is

$$\vec{P}(t) = -e \vec{x}(t) n_b = \frac{e^2 n_b}{m \omega_0^2} \vec{E}(t) = \chi_E \vec{E}(t)$$

where n_b is the density of bound electrons.

$$\epsilon = 1 + 4\pi \chi_E$$

B

*Could you turn this into a quantitative model of the dielectric constant, for low frequencies?
 $\hbar \omega_0 \sim \Delta E$ for the molecular energy levels.*

5✓

■ CONDUCTIVITY (PLASMAS AND METALS)

For free electrons, $\omega_0 = 0$.

Now the *model velocity of the negative charge* is

$$\begin{aligned} v_x(t) &= \dot{x}(t) \\ &= -i\omega (-(e/m) \mathcal{E}(\omega)) / (-\omega^2 - i\gamma\omega) e^{-i\omega t} \\ &= \frac{-e \mathcal{E}(\omega)}{m(\gamma - i\omega)} e^{-i\omega t} \end{aligned}$$

And so the current density is

$$J_x(t) = n_c (-e) v_x(t) = \sigma(\omega) E_x(t)$$

where n_c = density of free electrons (or, conduction electrons in a metal);

$$\sigma(\omega) = \text{conductivity} = \frac{n_c e^2}{m(\gamma - i\omega)}$$

Eq. 9.120

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Comments

- $\vec{J} = \sigma \vec{E}$ is Ohm's law.
- conductivity $\propto 1/\gamma$; \therefore resistivity $\equiv 1/\text{conductivity} \propto \gamma$ (resistance to motion)
- $\sigma = \sigma(\omega)$ for a harmonic field; and σ is complex;
- $\sigma(\omega) = n_c e^2 / [m(\gamma - i\omega)]$
- The real part of $\sigma(\omega)$ is not everything!
We don't just want the real part of σ ; but we do want the real part of \vec{J} .

■ WT (page ??)

"The complex conductivity produces a phase shift between the current oscillations and the electric field oscillations."

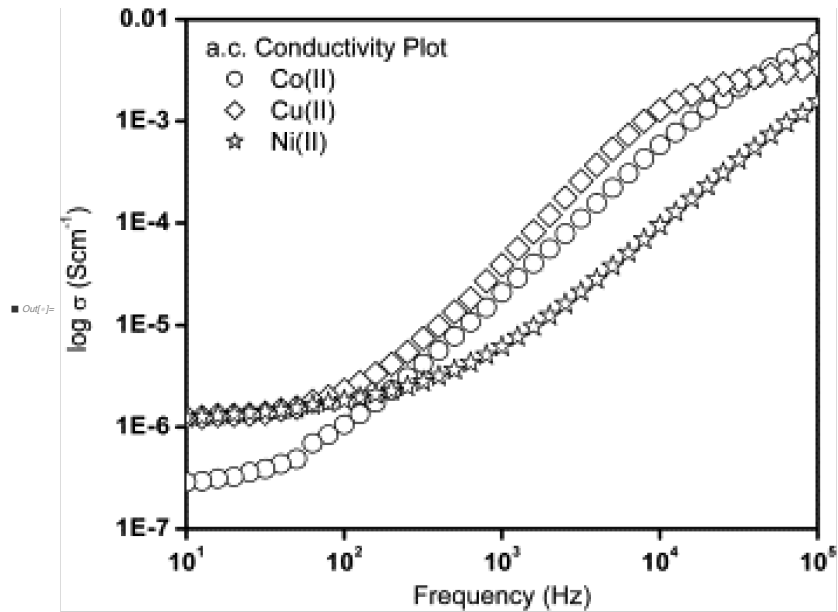
$$\begin{aligned} J_x(t) &= \text{Re} \{ \sigma(\omega) \mathcal{E}(\omega) e^{-i\omega t} \} \\ &= \dots \text{a few lines of algebra} \dots \\ &= \frac{n_c e^2}{m \sqrt{\gamma^2 + \omega^2}} \text{Re} \{ \mathcal{E}(\omega) e^{-i(\omega t + \delta)} \} \end{aligned}$$

where $\tan \delta = \frac{\omega}{\gamma}$.

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Example.

This figure shows the conductivity of metals at low frequencies (copied from some unknown web site)



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$$\text{Model : } \vec{x}(t) = \frac{-e \mathcal{E}(\omega)/m}{\omega_0^2 - \omega^2 - i \gamma \omega} e^{-i \omega t}$$

(Re implied)

PERMITTIVITY (RETURNING TO DIELECTRICS)

For bound electrons we have

$$\vec{x}(t) = \frac{e}{m} \text{Re} \left\{ \frac{-\vec{\mathcal{E}}(\omega) e^{-i \omega t}}{\omega_0^2 - \omega^2 - i \gamma \omega} \right\}$$

The polarization $\vec{P}(\vec{\xi}, t)$:

$$\vec{P}(\vec{\xi}, t) = n_{\text{bound}}(\vec{\xi}) \vec{p}(\vec{\xi}, t) = n_b(\vec{\xi}) (-e) \vec{x}(t)$$

$$\vec{P}(\vec{\xi}, t) = n_b(\vec{\xi}) e (e/m) \frac{\vec{\mathcal{E}}(\vec{\xi}, \omega) e^{-i \omega t}}{\omega_0^2 - \omega^2 - i \gamma \omega}$$

$$\vec{P}(\vec{\xi}, t) = \chi(\omega) \vec{E}(\vec{\xi}, t)$$

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The frequency-dependent susceptibility is

$$\chi(\omega) = \frac{n_b e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i \gamma \omega}$$

Eq. (9.133)

and the frequency-dependent permittivity is

$$\epsilon(\omega) = 1 + 4 \pi \chi(\omega) .$$

Then the displacement field is

$$\vec{D}(\vec{x}, t) = \epsilon(\omega) \vec{E}(\vec{x}, t)$$

*(!! only valid for a harmonic field, ω)
(Also, Re is implied.)*

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DISPERSION

The permittivity ϵ depends on the frequency $\omega/(2\pi)$.

Therefore Snell's law ($n_i \sin \theta_i = n_t \sin \theta_t$) implies that *refraction of light separates the colors by angles*.

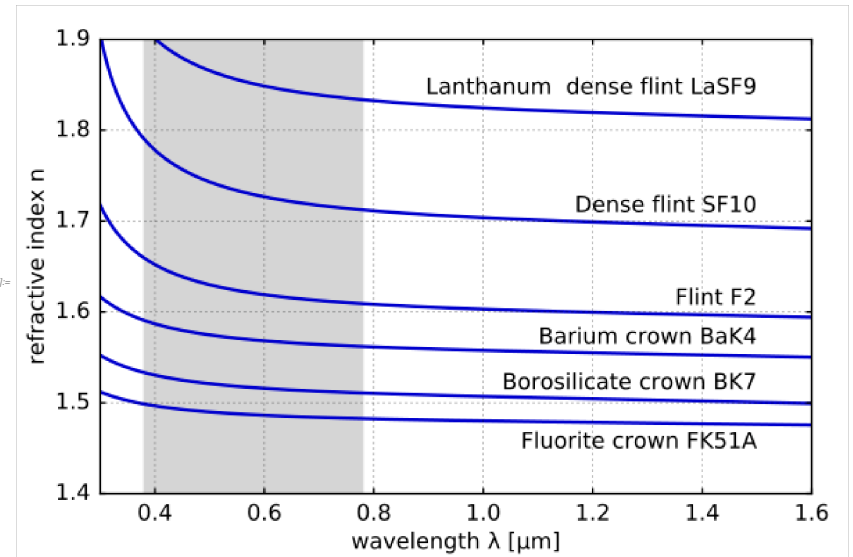
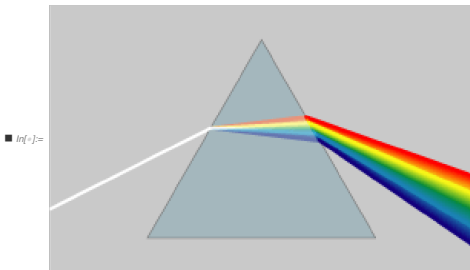
Consider sunlight (= white light ; a mixture of all the colors) incident from air ($n_i = 1$) into glass ($n_t \approx 1.50$). Suppose the angle of incidence is $\theta_i = 30$ degrees.

The angle of refraction will be $\theta_t \approx \arcsin(\sin 30 / n_t) = \arcsin(1/3) = 19.5$ degrees.

But that's not all, because n_t depends of ω ; $n =$

$$\sqrt{\epsilon(\omega) \mu(\omega)} \approx \sqrt{\epsilon(\omega)}.$$

Images from Wikipedia



$$n_{\text{glass}}(\text{red light}) < n_{\text{glass}}(\text{violet light})$$

$$\text{Range of } n_{\text{glass}} \approx 1.50 \sim 1.75$$

$$\text{Dispersion} = (1/n) (dn/d\lambda)$$

$$\approx (1/1.6) (-0.03 / 0.4 \mu\text{m}) \approx -0.05 / \mu\text{m}$$

(Range of visible light is $\Delta\lambda \approx 0.4 \mu\text{m}$.)

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Dispersion of white light passing through
droplets of water.

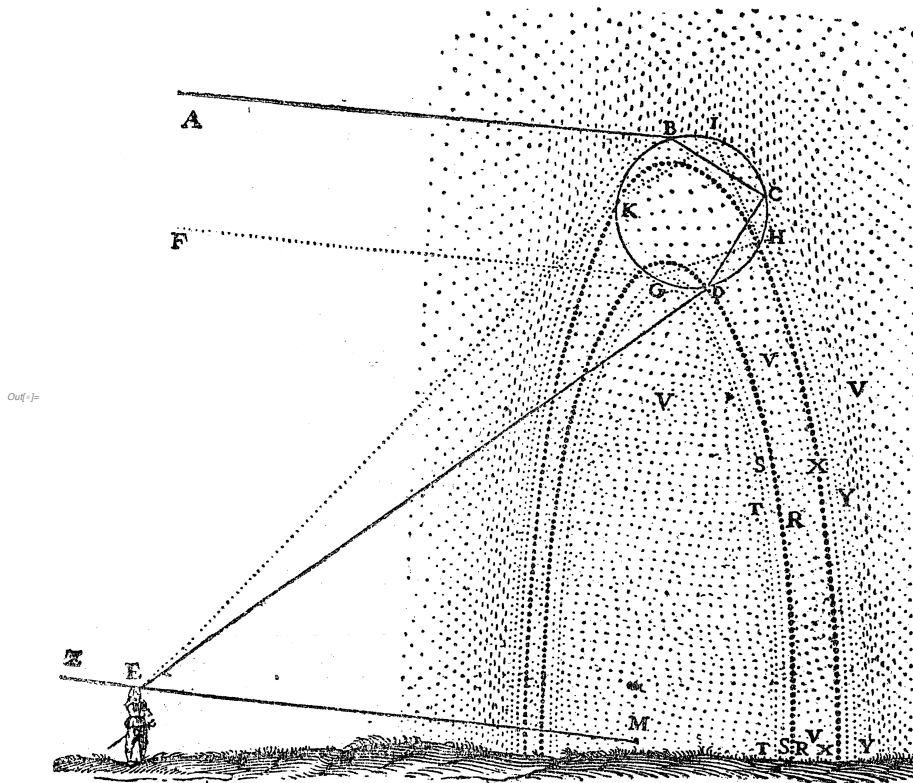
■ $f(x) =$



■ 9

m[-]= r1

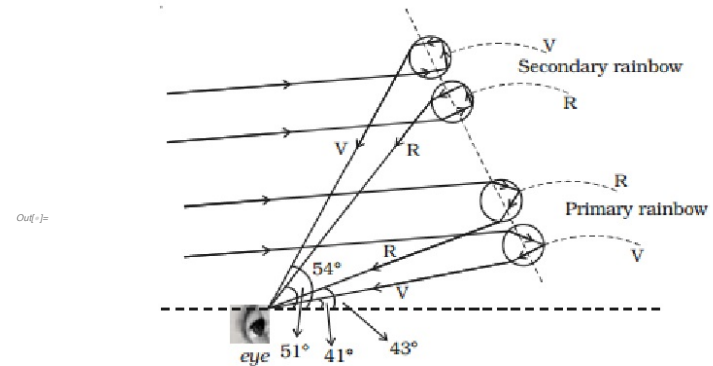
Rene DesCartes



Out[-]=

m[-]= r2

- Rene DesCartes analyzed and observed light incident on a dielectric sphere,
- processes of refraction, internal reflection, and a second refraction;
 - dispersion separates the colors;
 - the angle with maximum intensity (“caustic”) = rainbow angle = 40 - 42 degrees for water.



Out[-]=

Fig. Formation of rainbows