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# Chapter 9 - PLANE E. M. WAVES AND PROPAGATION IN MATTER

★ Lecture #4 on waves in matter

★ Section 9.6
*"Modeling plasmas, metals and dielectrics"*

## Combining Conduction and Polarization

$$m\,[\,\ddot{\vec{x}} + \gamma\,\dot{\vec{x}} + \omega_0^2\,\vec{x}\,] = -\,e\,\vec{E}(t)$$

Parameters:
$-e$ = "effective negative charge";
m = "effective mass" of the negative charge ;
$\gamma$ and $\omega_0$ .



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Plasmas, metals and dielectrics

Metals have both conduction electrons, for which $\omega_0 = 0$; and bound electrons—i.e., bound to ionic centers—for which $\omega_0 > 0$ . Then there exists both conductivity and permittivity.

Harmonic wave propagation depends on this Maxwell equation,

$$\nabla \times \vec{H} = (4\pi/c)\,\vec{J} + (1/c)\,\partial \vec{D}/\partial t$$

where

$$\vec{D} = \epsilon(\omega)\,\vec{\mathcal{E}}\,e^{i\left(\vec{k}\cdot\vec{x} - \omega t\right)} \quad \text{(Re implied; } \omega\text{)}$$

$$\vec{J} = \sigma(\omega)\,\vec{\mathcal{E}}\,e^{i\left(\vec{k}\cdot\vec{x} - \omega t\right)} \quad (\text{ " })$$

$$\therefore\ \nabla \times \vec{H} = [\,4\pi\,\sigma(\omega) - i\,\omega\,\epsilon(\omega)\,]/c \times \vec{\mathcal{E}}\,e^{i(k\cdot x - \omega t)}$$

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$$\boxed{\nabla \times \vec{H} = [\, 4\pi\, \sigma(\omega) \; - \mathrm{i}\, \omega\, \epsilon(\omega) \,]/\mathrm{c} \;\; \vec{\mathcal{E}}\, e^{i\,(k\cdot x\, -\, \omega t)}}$$

We could write the equation in terms of an effective conductivity,

$$\nabla \times \vec{H} = (4\pi/\mathrm{c})\, \sigma_{\mathrm{eff}}(\omega)\;\; \vec{\mathcal{E}}\, e^{i\left(\vec{k}\bullet\vec{x}\, -\, \omega t\right)}$$

where

$$\sigma_{\mathrm{eff}}(\omega) = \sigma(\omega) - \mathrm{i}\, \omega/(4\pi)\, \epsilon(\omega) \;\; ;$$

or an *effective permittivity*,

$$\nabla \times \vec{H} = (-\mathrm{i}\omega/\mathrm{c})\, \epsilon_{\mathrm{eff}}(\omega)\;\; \vec{\mathcal{E}}\, e^{i\left(\vec{k}\bullet\vec{x}\, -\, \omega t\right)}$$

where

$$\epsilon_{\mathrm{eff}}(\omega) \; = \; \epsilon(\omega) \; + \; (\, 4\, \pi\, i\, /\, \omega\, )\, \sigma(\omega)\, .$$

We'll use the latter approach.



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## The Dispersion Relation for E. M. waves

Recall the elementary result from Section 9.1,

$$c^2\, k^2 = \epsilon\, \mu\, \omega^2;$$

i.e., $\mathrm{n} = \mathrm{c}/\, v_{\mathrm{phase}} = \mathrm{ck}/\omega = \sqrt{\epsilon\mu}$ .

But now we'll have, for harmonic wave $\omega$,

$$c^2\, k^2 = \mu\, \omega^2\, [\, \epsilon(\omega) \; + \; 4\, \pi\, i\, \sigma(\omega)\, /\, \omega\, ].$$

[ Exercise 9.5.5 ]

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*Section 9.6*
*Modeling plasmas, metals and dielectrics*

So, most generally, we can write

$$\epsilon_{\text{eff}}(\omega) = \epsilon_{\text{bound}}(\omega) + 4\pi \, \text{i} \, \sigma_{\text{conduction}}(\omega)/\omega$$

***From now on we'll just write $\epsilon(\omega)$ to denote $\epsilon_{eff}(\omega)$.***

$\Longrightarrow$ Equation (9.148)

$$\epsilon(\omega) = 1 + \frac{4\pi e^2}{m}\{\Sigma \text{ over } i\} \frac{n_{b,i}}{\omega_{0,i}^2 - \omega^2 - i\,\omega\,\gamma_i} + \frac{4\pi i}{\omega} \; \frac{e^2 n_c}{m} \; \frac{1}{\gamma - i\,\omega}$$



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## Low frequencies in a metal $\Longrightarrow$ Skin Depth

Suppose

$\omega \ll \omega_0$ and $\omega \ll \gamma$ and $\omega \ll \sigma(0)$ ;

then we can approximate

$$\epsilon(\omega) \approx \epsilon_0 + 4\pi \text{i} \, \sigma(0)/\omega \approx 4\pi \text{i} \, \sigma(0)/\omega$$

$$\therefore \quad c^2 k^2 = \omega^2 \mu \, \epsilon \approx \omega^2 \mu \, [\, 4\pi \, \text{i} \, \sigma(0)/\omega \,].$$

*Exercise : Derive equations for $\epsilon_0$ and $\sigma(0)$*

Then,

$$ck \approx (1+\text{i})\sqrt{2\pi\,\mu\,\sigma(0)\,\omega} \equiv (1+\text{i})\, c/\delta(\omega) ,$$

where

$$\delta(\omega) = \frac{c}{\sqrt{2\pi\,\mu\,\sigma(0)\,\omega}}$$

$\delta(\omega)$ is called the skin depth. (Check the units!)

*The e.m. wave is damped exponentially* with characteristic damping distance $\delta$.

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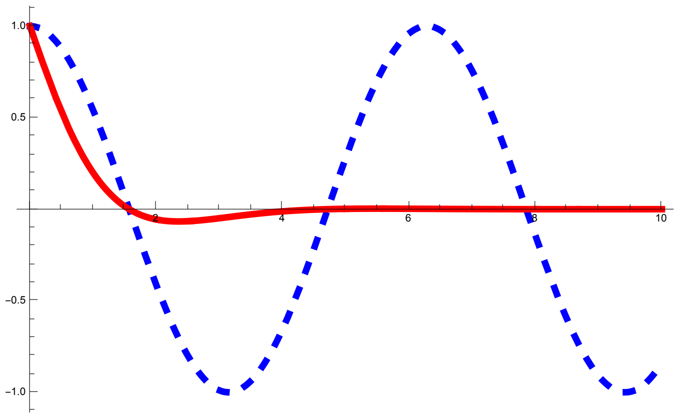
# The damped wave

In[9]:=
```
δ = 1;
Plot[{Cos[x / δ], Re[Exp[I * x / δ - x / δ]]}, {x, 0, 10},
 PlotRange → {All, All},
 PlotStyle →
  {{Dashing[{0.01, 0.03}], Thickness[0.01], Blue},
   {Thickness[0.01], Red}},
 ImageSize → Large]
```

Out[10]=





Damping of low frequency e.m. waves is typical in metals.

“Low frequency” will include IR waves, and even visible light.

*That is why metals are opaque.*

*And that is why metals are shiny.*

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## High frequencies ; say UV light and beyond

Consider dielectrics with $\omega \gg \omega_0$ , or conductors (plasmas or metals) with $\omega \gg \gamma$.

For these cases we can approximate

$$\epsilon(\omega) \approx 1 - \frac{4\pi e^2 N}{m\,\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

where $\omega_p^2 = 4\pi e^2 \mathrm{N} / \mathrm{m}$ ;

$\omega_p$ is called the *plasma frequency* .

NOTE: $N = n_b + n_c$ is the total density of electrons (bound + conduction). *The dielectric constant ( ≡ effective permittivity) is less than 1, and depends on* $\omega$.



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$$\boxed{\epsilon\,(\omega) = 1 - \omega_p^2\;/\,\omega^2}$$

## Dielectrics at high frequencies

$\omega \gg \omega_0$ normally implies $\omega \gg \omega_p$ so $\epsilon(\omega)$ is slightly less than 1, and the index of refraction is slightly less than 1. This is not familiar; for *classical optics* the index of refraction is greater than 1. Recall $v_{\text{phase}}$ = c/n. Here $v_{\text{phase}}$ > c. (!)

## Conductors (metals and plasmas) at high frequencies

$\omega_0 = 0$ and $\omega \gg \gamma$.

***But $\omega$ may be < $\omega_p$ .***

Then $\epsilon(\omega) = 1 - \omega_p^2\;/\,\omega^2$ can be $\ll 1$ or even ***negative***. (!)

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## THE DISPERSION RELATION FOR WAVES IN A CONDUCTOR AT HIGH FREQUENCIES; E.G., A PLASMA.

The dispersion relation for EM waves in the material becomes strongly $\omega$ dependent,

$$c^2 k^2 = \epsilon \mu \omega^2 = \omega^2 - \omega_p^2 \quad (\mu = 1)$$

### For $\omega < \omega_p$ ,

$$ck = i \sqrt{\omega_p^2 - \omega^2}$$

so the waves are damped ( b/c k is imaginary). E.g., microwaves cannot pass through a metal wall. Your cell phone may not work inside a building with metal construction.
Also, AM radio waves (535 - 1605 kHz) reflect from the ionosphere. Sky wave propagation.



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### For $\omega > \omega_p$ ,

$$ck = \sqrt{\omega^2 - {\omega_p}^2}$$

so the waves propagate in the material ( b/c k is real). For example, FM waves (88 - 108 MHz) propagate through the ionosphere.
Note that *the phase velocity is greater than the speed of light*,

$$v_{\text{phase}} = \omega / k = c\, \omega \Big/ \sqrt{\omega^2 - \omega_p^2} \; > \; c \, .$$

Does that violate the theory of relativity?

*(False statement: "Nothing can travel faster than the speed of light.")*

The *group velocity* is less than the speed of light,

$$v_{\text{group}} = \frac{d\omega}{dk} = c \sqrt{1 - \omega_p^2 / \omega^2} \; < \; c$$

$$v_{\text{group}} * v_{\text{phase}} = c^2$$

*(True statement : "No signal can travel faster than the speed of light.")*