0–

Quick review of last week --

Complex Permittivity

$\epsilon(\omega) = 1 + (4\pi e^2/m)\, n_e\, [\, \omega_0^2 - \omega^2 - i\,\gamma\,\omega\,]^{-1}$

Dielectric Response

$G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, e^{-i\omega'\tau}\, [\, \epsilon_{\text{eff}}(\omega') - 1\,]$

Reading Assignment:
Wikipedia article "Permittivity"
Section 6 - Dispersion and Causality

1-

Section 9.8
"Dispersion - a 1D example"

Given $\epsilon(\omega)$ and $\mu(\omega)$,
the *harmonic* electric field (and magnetic field) of an electromagnetic wave in a continuous medium obeys the wave equation,

$$\left\{ \nabla^2 - \frac{\mu\,\epsilon}{c^2} \frac{\partial^2}{\partial t^2} \right\} \vec{E} = 0 \tag{9.202}$$

$\Longrightarrow$ the phase velocity is

$$v_{\text{phase}} = \frac{\omega}{k} = \frac{c}{\sqrt{\mu\,\epsilon}} \tag{9.203}$$

At "low frequencies", $\epsilon > 1$ and then $v_{\text{phase}} < c$.

−1

But we have seen cases where $\epsilon(\omega)$ has the form

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

which implies $v_{\text{phase}} > c$.

Does that contradict the theory of special relativity?

(obviously, no; but why not?)

$v_{group} > c$ and $v_{phase} \times v_{group} = c^2$

2–

WAVE PACKETS, PLANE WAVES, PHASE VELOCITY AND GROUP VELOCITY

Go back to Maxwell's equations

$$\nabla \times \vec{H} = \frac{1}{c}\dot{\vec{D}} + \frac{4\pi}{c}\vec{J}_{\text{free}}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\dot{\vec{B}}$$

Considering waves in the material, $\vec{J}_{\text{free}} = 0$; and then recall from Friday,

$$\vec{D}(\vec{x}, t) = \vec{E}(\vec{x}, t) + \int_{0-}^{\infty} d\tau\, \vec{E}(\vec{x}, t-\tau)\, G(\tau)$$

$$G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, e^{-i\omega'\tau}\, [\, \epsilon_{\text{eff}}(\omega') - 1\,]$$

–2

Maxwell's equations become *(also, set $\mu = 1$)*

$$\nabla \times \vec{B} = (\text{displacement current})$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{1}{c}\int_{0-}^{\infty} d\tau\, \frac{\partial}{\partial t}\vec{E}(\vec{x}, t-\tau)\, G(\tau)$$

(a)

and $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{B}$ *(b)*

We can combine *(a)* and *(b)* to derive a new wave equation,

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{1}{c^2}\int_{0-}^{\infty} d\tau\, \frac{\partial^2}{\partial t^2}\vec{E}(\vec{x}, t-\tau)\, G(\tau)$$

This new wave equation is nonlocal in time. Causality.

3−

FOR SIMPLICITY, REDUCE THE DISCUSSION

TO A 1D PROBLEM WITH SIMILAR FEATURES

u(x,t)

⟸ so the rest is just math,
not electromagnetism;
"analogue problem"

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2}\int_{0-}^{\infty} d\tau\, \frac{\partial^2}{\partial t^2} u(x, t-\tau)\; G(\tau)$$

Analyze this analogue problem by Fourier analysis.

<u>FIRST,</u> transform from time (t) to frequency (ω).

−3

In[659]:= scanW61

Out[659]=

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\left\{ A(\omega)\, e^{ik(\omega)x} + B(\omega)\, e^{-ik(\omega)x} \right\}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{2\pi}}\int d\omega\, e^{-i\omega t}\left(-k^2(\omega)\right)\{\ldots\}$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{\sqrt{2\pi}}\int d\omega\, e^{-i\omega t}\left(-\frac{\omega^2}{c^2}\right)\{\ldots\}$$

$$\frac{1}{c^2}\int_{0-}^{\infty} d\tau\, \frac{\partial^2}{\partial t^2} u(x, t-\tau)\, G(\tau)$$

$$\hookrightarrow \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi} e^{-i\omega'\tau}\left[\epsilon(\omega')-1\right] = -\delta(\tau) + \int_{-\infty}^{\infty}\frac{d\omega'}{2\pi} e^{-i\omega'\tau}\epsilon(\omega')$$

$$\hookrightarrow\; = \frac{-1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega'\, e^{-i\omega' t}\left[-\epsilon(\omega')\frac{\omega'^2}{c^2}\right]\{\ldots\}$$

The PDE requires

$$k^2(\omega) = \epsilon(\omega)\frac{\omega^2}{c^2} \;\Rightarrow\; k(\omega) = \pm\, n(\omega)\frac{\omega}{c}.$$

$A(\omega)$ and $B(\omega)$ would be calculated from boundary conditions at x = 0.

4−

<u>SECOND,</u> transform from space (x) to reciprocal space (k).

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scanW63
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$$u(x,t) = \mathrm{Re}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{-i\omega(k)t}\left\{A(k)\,e^{ikx}\right\}$$

$$\frac{\partial^2 u}{\partial x^2} = \mathrm{Re}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{-i\omega t}\left(-k^2\right)A(k)\,e^{ikx}$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \mathrm{Re}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{-i\omega t}\left(\frac{-\omega^2}{c^2}\right)A(k)\,e^{ikx}$$

$$\frac{1}{c^2}\int_0^{\infty} d\tau\,\frac{\partial^2}{\partial t^2}\,u(x,t-\tau)\,G(\tau) = \frac{-1}{c^2}\frac{\partial^2 u}{\partial t^2} + \mathrm{Re}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{-i\omega t}\left[-\epsilon\frac{\omega^2}{c^2}\right]A(k)\,e^{ikx}$$

So to solve the PDE we need

$$k^2 = \epsilon(\omega(k))\,\frac{\omega^2(k)}{c^2}.$$

−4

A(k) (complex) would be calculated from boundary conditions at t = 0.

– ■ –

The dispersion relation––relating ω and k–– is, as usual,

$$k^2 = \epsilon_{\mathrm{eff}}(\omega)\,\omega^2\;/\,c^2$$

Also, WT deduce some other analytic properties of the solutions,

- u(x,t) should be real ;
- $n^*(-\omega) = n(\omega)$;
- $\omega^*(-k) = \omega(k)$.

5−

Consider a Quadratic dispersion relation

So far the analysis has been general.
Now specialize to a quadratic dispersion relation,

$$\omega(\mathrm{k}) = \nu \left(1 + \frac{a^2 k^2}{2}\right)$$

Or, since $k^2 = \epsilon\, \omega^2 / c^2$, ... a little algebra gives

...

$$\epsilon(\omega) = \frac{2\, c^2}{a^2\, \omega} \left(\frac{1}{\nu} - \frac{1}{\omega}\right).$$

(we're taking $\mu = 1$)

−5

So, the 1D analogue problem has

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2} \int_{0-}^{\infty} d\tau \frac{\partial^2}{\partial t^2} \mathrm{u}(\mathrm{x}, \mathrm{t} - \tau)\, \mathrm{G}(\tau)$$

(9.232)

and

$$\mathrm{G}(\tau) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, e^{-i\,\omega'\,\tau}\, [\, \epsilon(\omega') - 1\,]$$

(9.233)

with $\epsilon(\omega) = \frac{2\, c^2}{a^2\, \omega} \left(\frac{1}{\nu} - \frac{1}{\omega}\right)$

WT point out some problems with these equations (page 462). But then they say "let's just plow ahead and see what we get." After all, it is only an analogue problem, not electromagnetism.

6–

In[662]:= **scanW64**

Out[662]=

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2}\int_{0-}^{\infty} d\tau\, \frac{\partial^2}{\partial t^2} u(x, t-\tau)\, G(\tau)$$

$\partial/\partial t = -\partial/\partial\tau$; integrate by parts

$$= \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2}\int_{0-}^{\infty} d\tau\, u(x, t-\tau)\, \frac{d^2 G}{d\tau^2}$$

$$\frac{d^2 G(\tau)}{d\tau^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau} \left[\frac{2c^2}{a^2}\left(1-\frac{i\omega}{i\nu}\right) + \omega^2\right]$$

$$= \frac{2c^2}{a^2}\left[\delta(\tau) + \frac{1}{i\nu}\delta'(\tau)\right] - \delta''(\tau)$$

–6

In[667]:= **scanW65**

Out[667]=

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{c^2}\int_{0-}^{\infty} d\tau\, u(x, t-\tau) \left[\frac{2c^2}{a^2}\delta(\tau) + \frac{2c^2}{a^2 i\nu}\delta'(\tau) - \delta''(\tau)\right]$$

$$= \frac{2}{a^2}\left[u - \frac{i}{\nu}\frac{\partial u}{\partial t}\right](x,t)$$

Let $y(x,t) = u(x,t)\, e^{i\nu t}$

$$\Rightarrow \quad -\frac{\nu a^2}{2}\frac{\partial^2 y}{\partial x^2} = i\frac{\partial y}{\partial t}$$

7–

What is the result?

It's the Schroedinger equation!

$$-\frac{\hbar^2}{2m}\frac{\partial^2 y}{\partial x^2} = i\,\hbar\,\frac{\partial y}{\partial t}$$

$$\frac{\hbar}{2m} = \frac{v\,a^2}{2}$$

.

You know all about *wave packet solutions* of the Schroedinger equation, from courses on quantum mechanics. For example, Figure 9.12. You can read the rest of Section 9.9. Also, do Exercise 9.8.1 (homework assignment #5).

–7

Figure 9.12 - Real part and envelope at t = 0. How does it evolve in time?

In[665]:= Fig912

Out[665]=