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Section 9.9
Charged particle energy loss in materials

We'll use our model for the effective dielectric constant $\epsilon_{\text{eff}}(\omega)$ to get an understanding of *dissipative energy loss* by a charged particle (mass M) moving through the material.

For nonrelativistic particle energies, we can have

- *electron excitations* in the molecules
- *ionization* of the molecules

(For higher energies there would also be radiative energy loss, e.g., from Bremsstrahlung as the particle scatters from nuclei; Chapter 11.)

A particle on a straight line path

The particle (M) could be a sufficiently high-energy electron ($M = m_{\text{electron}}$), or a muon or meson ($M > 100\, m_{\text{electron}}$) or a

light ion ($M > 2000\, m_{\text{electron}}$).

The particle interacts with the electrons of the material—both bound and conduction electrons.

The angular deflections of the particle are are small.

Therefore the free charge and current densities are

$$\rho(\vec{x},t) = q\, \delta^3(\vec{x} - \vec{v}\, t)$$

$$\vec{J}(\vec{x},t) = q\, \vec{v}\, \delta^3(\vec{x} - \vec{v}\, t)$$

.

q = particle charge;

$\vec{v}$ = velocity vector, assumed to be constant.

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In frequency space, and the reciprocal of position space,

$$\rho\,(\vec{k},\omega) \equiv \frac{1}{(2\,\pi)^2} \int d^3x \int dt\;\; \rho\,(\,\vec{x}\,,t\,)\times$$

$$\times\exp\{\,-i\,(\,\vec{k}\cdot\vec{x} - \omega\,t)\,\}$$

$$= \frac{q}{2\,\pi}\;\delta\,(\omega - \vec{k}\;.\;\vec{v}\,)$$

.

$$\vec{J}(\,\vec{k},\omega) \equiv \frac{1}{(2\,\pi)^2} \int d^3x \int dt\;\; \vec{J}(\,\vec{x}\,,t\,)\times$$

$$\times\exp\{\,-i\,(\,\vec{k}\cdot\vec{x} - \omega\,t)\,\}$$

$$= \frac{q}{2\,\pi}\;\vec{v}\;\delta\,(\omega - \vec{k}\;.\;\vec{v}\,)$$

The electric field

$$\nabla\cdot\vec{D}\,(\,\vec{x},\,t) = 4\pi\,\rho\,(\,\vec{x},\,t)$$

$$i\,\vec{k}\cdot\vec{D}\,(\,\vec{k},\,\omega\,) = 4\pi\,\rho\,(\,\vec{k},\omega\,)$$

$$\vec{D}(\,\vec{k},\,\omega\,) = \epsilon_{\mathrm{eff}}(\omega)\;\vec{E}(\,\vec{k},\,\omega\,)$$

As before, $\epsilon_{\mathrm{eff}}(\omega) = \epsilon(\omega) + \dfrac{4\,\pi\,i\,\sigma(\omega)}{\gamma - i\,\omega}$

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<u>**The Potential**</u>

In general $\vec{E} = -\nabla\,\Phi\; - (1/c)\;\partial\vec{A}\,/\partial t$.
For nonrelativistic energies we can neglect the vector potential.

The trajectory of the particle is determined mainly by the Coulomb force; magnetic interactions would be important for a relativistic particle, v⟶ c.

So,

$$\vec{E}(\,\vec{x},\,t\,) = -\nabla\;\Phi(\,\vec{x},t)$$

$$\vec{E}(\,\vec{k},\,\omega\,) = -\,i\,\vec{k}\;\Phi(\,\vec{k},\,\omega\,)$$

In[775]:= `scanW71`

Out[775]=

Calculation

$$i\vec{k}\cdot\vec{D}(\vec{k},\omega) = 4\pi\rho(\vec{k},\omega)$$

$$LHS = i\vec{k}\cdot\epsilon_{eff}(\omega)\vec{E}(\vec{k},\omega)$$

$$= i\vec{k}\cdot\epsilon_{eff}(\omega)(-i\vec{k})\Phi(\vec{k},\omega)$$

$$= k^2\epsilon_{eff}(\omega)\Phi(\vec{k},\omega)$$

$$RHS = 4\pi\frac{q}{2\pi}\delta(\omega-\vec{k}\cdot\vec{v})$$

$$\therefore\ \Phi(\vec{k},\omega) = \frac{2q}{k^2\epsilon_{eff}(\omega)}\delta(\omega-\vec{k}\cdot\vec{v})$$

$$\therefore\ \vec{E}(\vec{k},\omega) = \frac{-2iq\vec{k}}{k^2\epsilon_{eff}(\omega)}\delta(\omega-\vec{k}\cdot\vec{v})$$

(256)

Thus,

$$\vec{E}(\vec{k},\omega) = \frac{-2iq\vec{k}}{k^2\epsilon_{\text{eff}}(\omega)}\delta(\omega-\vec{k}.\vec{v})$$

(256)

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Stopping Power

The particle loses energy as it moves through the material.

The "stopping power" is defined as dE/dx where E = energy of the particle (not electric field)!)

Recall conservation of energy,

$$\frac{dE}{dx} = \frac{1}{v}\frac{dE}{dt}$$

$$= -\frac{1}{v}\int d^3x\,\vec{J}(\vec{x},t)\cdot\vec{E}(\vec{x},t)$$

.

Note the minus sign. The particle does work on the sample,so the particle energy decreases.

In[762]:= **scanW72**

Out[762]=

Calculation

$$\frac{dE}{dx} = \frac{1}{v}\frac{dE}{dt} = \frac{-1}{v}\int d^3x\, \vec{J}(\vec{x},t)\cdot\vec{E}(\vec{x},t)$$

$$= \frac{-1}{v}\int d^3k\, \vec{J}(-\vec{k},t)\cdot\vec{E}(\vec{k},t) \qquad \text{Parseval's th.}$$

$$= \frac{-1}{2\pi v}\int d^3k \int d\omega \int d\omega'\, e^{-i\omega t}\, e^{-i\omega' t}\, \underbrace{\vec{J}(-\vec{k},\omega')}_{\frac{q\vec{v}}{2\pi}\delta(\omega'+\vec{k}\cdot\vec{v})} \cdot \underbrace{\vec{E}(\vec{k},\omega)}_{\left(\frac{-2iq\vec{k}}{k^2\epsilon_{eff}}\right)\delta(\omega-\vec{k}\cdot\vec{v})}$$

$$= \frac{2iq^2}{(2\pi)^2 v}\int \frac{d^3k}{k^2}\, \frac{\vec{k}\cdot\vec{v}}{\epsilon_{eff}(\vec{k}\cdot\vec{v})} \qquad (259)$$

In[768]:= **scanW73**

Out[768]=

Some results from Sec. 9.7

Write $\epsilon_{eff}(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega)$

Then $\epsilon_1(-\omega) = +\epsilon_1(\omega)$

and $\epsilon_2(-\omega) = -\epsilon_2(\omega)$

$$\int \frac{d^3k}{k^2}\frac{\vec{k}\cdot\vec{v}}{\epsilon_{eff}(\vec{k}\cdot\vec{v})} = \frac{1}{2}\int \frac{d^3k}{k^2}\frac{\vec{k}\cdot\vec{v}}{\epsilon_{eff}(\vec{k}\cdot\vec{v})} + \frac{1}{2}\int\frac{d^3k}{k^2}\frac{(-\vec{k}\cdot\vec{v})}{\epsilon_{eff}(-\vec{k}\cdot\vec{v})}$$

$$= \frac{1}{2}\int\frac{d^3k}{k^2}\frac{\vec{k}\cdot\vec{v}}{\epsilon_1^2+\epsilon_2^2}\Big\{\epsilon_1(\vec{k}\cdot\vec{v}) - i\epsilon_2(\vec{k}\cdot\vec{v}) - \epsilon_1(-\vec{k}\cdot\vec{v}) + i\epsilon_2(-\vec{k}\cdot\vec{v})\Big\}$$

cancel add

$$= -i\int\frac{d^3k}{k^2}\,\vec{k}\cdot\vec{v}\,(-)\,\mathrm{Im}\left[\frac{1}{\epsilon_{eff}(\vec{k}\cdot\vec{v})}\right]$$

In[769]:= `scanW74`

Out[769]=

So

$$\frac{dE}{dx} = \frac{2i\,q^2}{(2\pi)^2 v}\, i\int \frac{d^3k}{k^2}\, \vec{k}\cdot\vec{v}\; \mathrm{Im}\left[\frac{1}{\epsilon_{eff}(\vec{k}\cdot\vec{v})}\right]$$

$$\frac{dE}{dx} = \frac{-q^2}{2\pi^2 v}\int \frac{d^3k}{k^2}\, \vec{k}\cdot\vec{v}\; \mathrm{Im}\left[\frac{1}{\epsilon_{eff}(\vec{k}\cdot\vec{v})}\right] \qquad (261)$$

WLOG Let $\vec{v} = v\hat{e}_z$.

$$d^3k = d^2k_\perp\, dk_z = 2\pi\, k_\perp dk_\perp\, dk_z$$

Let $\omega = \vec{k}\cdot\vec{v} = v k_z \Rightarrow dk_z = \frac{d\omega}{v}$

$$\frac{dE}{dx} = \frac{-q^2}{2\pi^2 v}\, 2\pi \int \frac{k_\perp dk_\perp\, d\omega/v}{k_\perp^2 + (\omega/v)^2}\,\omega\, \mathrm{Im}\left[\frac{1}{\epsilon_{eff}(\omega)}\right]$$

$$= \frac{-q^2}{\pi v^2}\int \frac{k_\perp dk_\perp}{k_\perp^2 + (\omega/v)^2}\, d\omega\, \omega\, \mathrm{Im}\left[\frac{1}{\epsilon_{eff}(\omega)}\right]$$

Next we need to integrate over the transverse component $k_\perp$.

If we integrate from 0 to ∞ the result is logarithmically divergent, because the integrand ~ $dk_\perp / k_\perp$ at large $k_\perp$.

But of course infinite $k_\perp$ makes no sense.

To proceed, we introduce a maximum value k_0 for $k_\perp$.

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<u>Semi-classical value of k_0.</u>

The particle has mass M, and it scatters from an electron with mass m.

The transverse momentum transfer is $\hbar k$.

So $\hbar k_0$ = the *maximum* transverse momentum transfer.

.

In the center of mass frame, M and m have momenta p_{cm} and $-p_{cm}$;

$$p_{cm} = \frac{Mm}{M+m}\, v$$

The maximum transverse momentum transfer occurs for scattering angle = 90 degrees, and is equal to 2 p_{cm}.

The result is,

$$\hbar k_0 = 2\, p_{cm} = \zeta\, m\, v$$

where $\zeta = 2/(1+m/M)$.

ζ = 1 for M = m ; ζ = 2 for M ≫ m.

Now complete the integration.

In[772]:= scanW75

Out[772]=

$$\int_0^{k_0} \frac{k_\perp dk_\perp}{k_\perp^2 + (\omega/v)^2} = \frac{1}{2} \ln\left[1 + \left(\frac{v k_0}{\omega}\right)^2\right]$$

$$\sim \frac{1}{2} \ln\left[1 + \left(\frac{\zeta m v^2}{\hbar \omega}\right)^2\right]$$

$$\sim \ln\left(\frac{\zeta m v^2}{\hbar \overline{\omega}}\right]$$

$$\frac{dE}{dx} = \frac{-q^2}{\pi v^2} \ln\left[\frac{\zeta m v^2}{\hbar \overline{\omega}}\right] \int_{-\infty}^{\infty} d\omega\, \omega\, \mathrm{Im}\left[\frac{1}{\epsilon_{eff}(\omega)}\right]$$

Result so far,

$$\frac{dE}{dx} = -\frac{q^2}{\pi v^2} \ln\left(\frac{\zeta\, m v^2}{\hbar\, \overline{\omega}}\right) \times$$

$$\times \int_{-\infty}^{\infty} \omega\, d\omega\, \mathrm{Im}\left(\frac{1}{\epsilon_{\mathrm{eff}}(\omega)}\right)$$

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The final step makes use of the Kramers Kronig relations (generalized)

In[776]:= scanW76

Out[776]=

Another sum rule

Recall, from K.-K. relations,

$$\omega_p^2 = \frac{2}{\pi}\int_0^\infty d\omega\, \omega\, \mathrm{Im}\, \epsilon_{eff}(\omega)$$

Now, $\epsilon_{eff}(\omega) = 1 - \omega_p^2/\omega^2$

$$\frac{1}{\epsilon_{eff}(\omega)} \sim 1 + \omega_p^2/\omega^2 \quad \text{for } \omega \gg \omega_p$$

$$\therefore \quad (\omega_p)^2_{eff} = -\frac{2}{\pi}\int_0^\infty d\omega\, \omega\, \mathrm{Im}\left[\frac{1}{\epsilon_{eff}(\omega)}\right]$$

scanW77

Out[777]=

$$\frac{dE}{dx} = \frac{q^2 (\omega_p)^2_{eff}}{v^2} \ln\left[\frac{\xi m v^2}{\hbar \bar{\omega}}\right]$$

$$(\omega_p)^2_{eff} = 4\pi (n_b + n_c)\frac{e^2}{m}$$

I dropped a factor of 1/2 here, to get agreement with Wilcox.

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Here is the final result in WT,

$$\frac{dE}{dx} = \frac{q^2}{v^2} \left(\omega_p\right)^2_{\text{eff}} \ln\left(\frac{\zeta\, m v^2}{\hbar\, \overline{\omega}}\right)$$

$$= \frac{4\,\pi\, N\, Z_1^2\, e^4}{m v^2} \ln\left(\frac{\zeta\, m v^2}{\hbar\, \overline{\omega}}\right) \qquad (272)$$

where N = total electron charge density,
Z_1e = q = charge of the particle,
v = velocity of the particle ($\ll$ c),
m = electron mass,
$\hbar\,\overline{\omega}$ = mean ionization energy along the path.

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Theory of stopping power

Section 9.10 "Going further"

- ***9.10.2 = Dissipative charged particle energy loss***
- ***References 1 - 6.***

Look up the Particle Data Group article entitled "Passage of Particles through Matter".

Assign a homework problem on Assignment #6.