1.

Maxwell's equations and potentials for time dependent sources

REVIEW ***A TOPIC FROM CHAPTER 8***

— POTENTIALS IN THE LORENZ GAUGE

Section 8.6: Retarded Potentials

Given $\rho(\vec{x}\,, t)$ and $\vec{J}(\vec{x}\,, t)$ with $\nabla\cdot\vec{J} = -\dfrac{\partial\rho}{\partial t}$

Maxwell's equations (1-4)

(1) $\nabla\cdot\vec{B} = 0 \implies \vec{B} = \nabla\times\vec{A}$

$\vec{A}\,(\vec{x}\,, t)$

(2) $\nabla\times\vec{E} + \dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t} = 0 \implies \vec{E} = \dfrac{-1}{c}\dfrac{\partial\vec{A}}{\partial t} - \nabla\Phi$

$\Phi\,(\vec{x}\,, t)$

We' ll also impose the gauge choice

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0\,.$$

LORENZ GAUGE CONDITION

.1

Maxwell's equations (3,4)

(3) $\nabla\cdot\vec{E} = 4\,\pi\,\rho$

$$= \nabla\cdot\left(\frac{-1}{c}\frac{\partial\vec{A}}{\partial t} - \nabla\Phi\right)$$

$$= +\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \nabla^2\Phi = \Box\,\Phi$$

D'Alebertian $\Box = \partial^2 / c^2\,\partial\,t^2 - \nabla^2$

$\therefore \quad \Box\,\Phi = 4\,\pi\,\rho\,(\vec{x}, t)$

In[99]:= .

(4) $\nabla\times\vec{B} - \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t} = \dfrac{4\,\pi}{c}\,\vec{J}$

$$= \nabla\times(\nabla\times\vec{A}) - \frac{1}{c}\left(\frac{-1}{c}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla\frac{\partial\Phi}{\partial t}\right) =$$

$\Box\,\vec{A}$

$\therefore \quad \Box\,\vec{A} = \dfrac{4\,\pi}{c}\,\vec{J}\,(\vec{x}, t)$

2.

THE GREEN'S FUNCTION OF THE D'ALEMBERTIAN

Define it like this,

$$\Box\, G(\vec{x},t\,;\vec{x}',t') = 4\,\pi\,\delta^3(\vec{x}-\vec{x}')\,\delta\,(t-t') \qquad (8.88)$$

Use Fourier analysis to solve it; $\Longleftarrow$ *WT equations 8.103 to 8.111 (reading assignment)*. I'll use a quicker method of derivation.

In[99]:= scanR01

Out[99]=

Write

$$G(\vec{x},t\,;\vec{x}',t') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \times G(\vec{k},\omega) \qquad (\star 1)$$

Plug that into the definition,

$$\Box\, G(\vec{x}\,t\,;\vec{x}'\,t') \qquad \text{where} \quad \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

$$= \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \times \left[-\omega^2/c^2 + \vec{k}^2\right] G(\vec{k},\omega)$$

$$= 4\pi\, \delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$$

$$= \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\; 4\pi$$

Thus

$$\left(\vec{k}^2 - \frac{\omega^2}{c^2}\right) G(\vec{k},\omega) = 4\pi$$

In[100]:= scanR02

Out[100]=

However, we can't use

$$G(\vec{k},\omega) = \frac{4\pi}{k^2-\omega^2/c^2} = \frac{4\pi c^2}{c^2k^2-\omega^2} \quad (NO!)$$

in equation (★1). Because the integrals would be undefined because of the singularities at $\omega = \pm ck$.

"Regularization"

Replace $\omega \to \omega + i\epsilon \Rightarrow \dfrac{4\pi c^2}{c^2k^2-(\omega+i\epsilon)^2}$

Here $\epsilon > 0$ and $\epsilon \longrightarrow 0$.

The retarded Green's function has $-i\epsilon$ in the denominator.

In[101]:= scanR03

Out[101]=

$$G(\vec{k},\omega) = \int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\int\frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \times \frac{4\pi c^2}{c^2k^2-(\omega+i\epsilon)^2}$$

$\left\{ \begin{array}{l} \omega \to \omega + i\epsilon \text{ gives the retarded Green's function} \\ \omega \to \omega - i\epsilon \text{ gives the advanced Green's function} \end{array} \right.$

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● Now do the integral over $\vec{k}$.

In[102]:= scanR04

Out[102]=

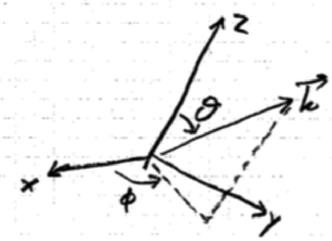
The retarded Green's function

$$G^{ret}(\vec{x}t;\vec{x}'t') = \int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\int\frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} \times \frac{4\pi c^2}{c^2k^2-(\omega+i\epsilon)^2}$$

The $\vec{k}$ integrals

WLOG define the z direction by $\vec{x}-\vec{x}'$.

- $d^3k = k^2\,dk\,\sin\theta\,d\theta\,d\phi$
- $\vec{k}\cdot(\vec{x}-\vec{x}') = kR\cos\theta$ where $R = |\vec{x}-\vec{x}'|$.
- $\int d\phi = 2\pi$



3.

In[103]:= `scanR05`

Out[103]=

$$J(\omega) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{c^2k^2-(\omega+i\epsilon)^2}$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty \frac{k^2\,dk}{c^2k^2-(\omega+i\epsilon)^2}\underbrace{\int_{-1}^{1} du\, e^{ikRu}}_{\frac{2}{kR}\sin(kR)}$$

$$= \frac{1}{2\pi^2 R}\int_0^\infty \frac{k\,dk}{c^2k^2-(\omega+i\epsilon)^2}\sin(kR)$$

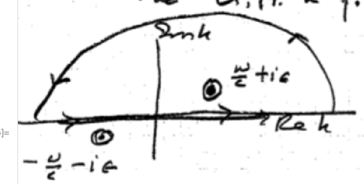
$$= \frac{1}{2\pi^2 R}\int_{-\infty}^{\infty} \frac{k\,dk}{c^2k^2-(\omega+i\epsilon)^2}\frac{1}{2i}e^{ikR}$$

In[104]:= `scanR0M`

Out[104]=

$$J(\omega) = \frac{1}{2\pi^2 R}\frac{1}{2i}\int_{-\infty}^{\infty}\frac{dk\, k\, e^{ikR}}{[ck-(\omega+i\epsilon)][ck+(\omega+i\epsilon)]}$$

Because $R>0$, close the contour in the U.H. k P.



$$\text{Residue} = 2\pi i\,\frac{(\frac{\omega}{c}+i\epsilon)\,e^{i(\frac{\omega}{c}+i\epsilon)R}}{\omega+i\epsilon+\omega+i\epsilon}$$

I don't need the regularization any more, so now set $\epsilon = 0$

$$J(\omega) = \frac{1}{2\pi^2 R}\frac{1}{2i}2\pi i\,\frac{\frac{\omega}{c}}{2\omega}\,e^{i\omega R/c}$$

$$J(\omega) = \frac{1}{4\pi R c}\,e^{i\omega R/c}$$

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● Now do the integral over ω.

In[105]:= `scanR07`

Out[105]=

The ω integral

$$G^{ret}(\vec{x}t;\vec{x}'t') = \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, 4\pi c^2 J(\omega)$$

$$= \frac{4\pi c^2}{4\pi R c} \int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t'-R/c)}$$

$$= \frac{c}{R}\, \delta(t-t'-R/c) \qquad (8.111)$$

$$G^{ret}(\vec{x},t\,;\vec{x}',t') = \frac{1}{R}\,\delta(t-t'-R/c)$$

where $R = |\vec{x}-\vec{x}'|$.

4.

$$G^{\text{ret}}(\vec{x},t\,;\vec{x}',t') = \frac{1}{R}\,\delta(t - t' - R/c)$$

where $R = |\vec{x} - \vec{x}'|$.

Interesting properties of the retarded Green's function

- Time reversal invariance,

$$G^{\text{ret}}(\vec{x}',-t'\,;\vec{x},-t) = G^{\text{ret}}(\vec{x},t\,;\vec{x}',t')$$

- Propagation time = R/c,

$G^{\text{ret}}(\vec{x},t\,;\vec{x}',t')$

= the field at $(\vec{x},t)$ due to a source at $(\vec{x}',t')$

= 0 unless $t = t' + |\vec{x} - \vec{x}'|/c$.

I.e., it takes time R/c for the signal to reach the point $(\vec{x}, t)$

.4

The Potentials

$$\Box\,\Phi = 4\pi\rho$$

$\Longrightarrow$

$$\Phi(\vec{x},t) = \int d^3x'\,dt'\;\; G^{\text{ret}}(\vec{x},t\,;\vec{x}',t')\;\;\rho(\vec{x}',t')$$

$\longleftarrow\!\!-$ $1/R\,\delta(t-t'-R/c)$ $-\!\!\longrightarrow$

$$\Phi(\vec{x},t) = \int d^3x'\;\frac{1}{R}\,\rho(\vec{x}',t-R/c)$$

where $R = |\vec{x} - \vec{x}'|$;

(11.1)

and similarly,

$$\vec{A}(\vec{x},t) = \int d^3x'\;\frac{1}{R}\,\vec{J}(\vec{x}',t-R/c)$$

(11.2)

Equations (11.1) and (11.2) are the equations for the potentials due to arbitrary time-dependent sources, in free space.

5.

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Out[106]=

The foundational equations for all of our radiation calculations in this chapter are the free-space retarded potentials in Lorenz gauge from (8.114) and (8.115). Recalling that $R \equiv |\vec{x} - \vec{x}'|$, we have:

In[107]:=
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Out[107]=

$$\Phi(\vec{x}, t) = \int d^3x' \, \frac{\rho(\vec{x}', t - R/c)}{R}, \tag{11.1}$$

$$\vec{A}(\vec{x}, t) = \frac{1}{c} \int d^3x' \, \frac{\vec{J}(\vec{x}', t - R/c)}{R}. \tag{11.2}$$

6.

See homework assignment #6.

Problem 6.3 : Exercise 8.6.1.

In[108]:=
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Out[108]=

Exercise 8.6.1. Find the solution of the free, one-dimensional wave equation Green function,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(x,t;x',t') = -4\pi\delta(x-x')\delta(t-t').$$

(the 4π is optional) with retarded time boundary conditions,

$$G(x,t;x',t') = 0 \quad \text{for } t < t'.$$

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Problem 6.4 : Exercise 8.6.4.

In[109]:=
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Out[109]=

Exercise 8.6.4.

(a) Show that the retarded three dimensional Green function,

$$G^{3D}(\vec{x},t;\vec{x}\,',t') = \frac{\delta(\tau - \frac{R}{c})}{R},$$

$(R = |\vec{x} - \vec{x}\,'|,\ \tau = t - t')$ satisfies

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G^{3D}(\vec{x},t;\vec{x}\,',t') = -4\pi\delta^{(3)}(\vec{x}-\vec{x}\,')\delta(t-t'),$$

directly by carrying out the specified derivatives. (You may choose $\vec{x}' = t' = 0$ for simplicity.)

(b) By using the integral form or by other means, prove (or argue) that

$$\lim_{(t-t')\to 0^+} G^{3D}(\vec{x},t;\vec{x}',t') = 0,$$

$$\lim_{(t-t')\to 0^+} \frac{\partial}{\partial t'} G^{3D}(\vec{x},t;\vec{x}',t') = -4\pi c^2\delta^{(3)}(\vec{x}-\vec{x}'),$$

where $\delta^{(3)}(\vec{x}-\vec{x}')$ is the three-dimensional spatial function. (The second result is the analog of the surface