1.

CHAPTER 11

Radiation by Systems and Point Particles

Lecture #1 on Radiation

Section 11.1
E. M. radiation by systems: the harmonic formalism

How are E.M. waves created?

11.1 – E. M. radiation by systems: the harmonic formalism

.1

The free-space retarded potentials, in the Lorenz gauge

Eqs (11.1) and (11.2)

$$\Phi(\vec{x}, t) = \int d^3x' \frac{1}{R} \rho(\vec{x}', t - R/c)$$

$$\vec{A}(\vec{x}, t) = \int d^3x' \frac{1}{cR} \vec{J}(\vec{x}', t - R/c)$$

where $R = | \vec{x} - \vec{x}' |$.

In[268]:= figF11p1

Out[268]=

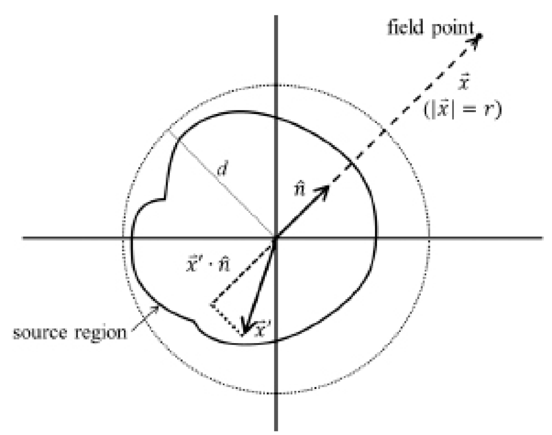

Fig. 11.1 Notation used in radiation calculations.

2.

Section 11.1 : The Harmonic Formalism

Assume the sources are harmonic in time $\Longrightarrow$ $e^{-i\omega t}$

The fields and potentials will have the same

harmonic time dependence.

Solve for $\vec{A}(\vec{x},t)$; also, the fields.
The results are only valid for *harmonic sources*. This is interesting on its own right.
And furthermore *any time dependence* can be written as a superposition of harmonic terms (Fourier analysis).

In[200]:= ·

Assume $\rho(\vec{x}, t) = \rho(\vec{x})\, e^{-i\omega t}$

and $\vec{J}(\vec{x}, t) = \vec{J}(\vec{x})\, e^{-i\omega t}$

(!! Re is understood.)

(!! $\rho(\vec{x})$ and $\vec{J}(\vec{x})$ may be complex functions.)

.2

(!! Be careful, because we are using the same symbol for two different things; e.g., $\vec{J}(\vec{x},t)$ and $\vec{J}(\vec{x})$.)

Then $\vec{A}(\vec{x},t) = \vec{A}(\vec{x})\, e^{-i\omega t}$

and $\Phi(\vec{x},t) = \Phi(\vec{x})\, e^{-i\omega t}$.

We will always use the __*Lorenz gauge*__. So, if we calculate $\vec{A}(\vec{x})$ then $\Phi(\vec{x})$ is immediately also known.

$\nabla \bullet \vec{A} + (1/c)\, \partial\Phi/\partial t = 0 \Longrightarrow \Phi = c/(i\omega)\, \nabla \bullet \vec{A}$

$\therefore$ We need to calculate ...

$$\vec{A}(\vec{x}, t) = \int d^3x' \, \frac{1}{c\,R} \, \vec{J}(\vec{x}', t - R/c)$$

where $R = |\, \vec{x} - \vec{x}'\, |$

3·

For a harmonic source,

$$\vec{A}(\vec{x}, t) = \int d^3x' \frac{1}{cR} \vec{J}(\vec{x}') e^{-i\omega(t - R/c)}$$

$$= \vec{A}(\vec{x}) e^{-i\omega t}$$

with

$$\vec{A}(\vec{x}) = \frac{1}{c} \int \frac{d^3x' \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} e^{i(\omega/c)|\vec{x} - \vec{x}'|}$$

We probably cannot calculate the integral exactly, so we'll use some approximations. The most interesting aspect of the fields is the *propagating wave*; i.e., the fields in the "far zone".

·3

In this problem there are three parameters with units of length.

- d = size of the radiating system
- λ = wavelength of the E.M. waves

We'll see that $\lambda = 2\pi c/\omega$.

- r = distance from the radiating system to the point where we are observing the fields

See Figure 11.1 .

In[281]:= figF11p1

Out[281]=

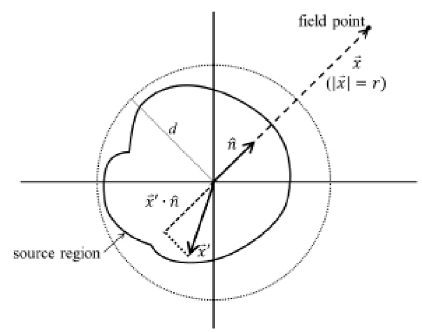

Fig. 11.1 Notation used in radiation calculations.

The "far zone" is r ≫ d and r ≫ λ.

4.

The "Near Zone" Approximation

Consider $d \ll r$ and $r \ll \lambda$.

Then $e^{ikR} \approx \exp\{ i\,(2\pi/\lambda)\, r \} \approx 1$.
So in the near zone,

$$\vec{A}(\vec{x}) \approx \frac{1}{c} \int d^3x' \; \frac{\vec{J}(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}$$

This is nothing but the "magnetostatic vector potential";

in other words, suppose $\vec{J}(\vec{x}\,', t') = \vec{J}(\vec{x}\,')$ (*i.e.*, $\omega = 0$).

.4

The "Far Zone" Approximation

Consider $r \longrightarrow \infty$; or, $r \gg d$ and $r \gg \lambda$.

The far zone is also called the "radiation zone".

These are the *asymptotic fields* — propagating away from the source.

Now, *what is* λ?

For now, we have defined $\lambda \equiv 2\pi\, c/\omega$. The dimension of λ is length. When we show that waves are propagating away from the source, then we'll see that λ is in fact the wavelength of the asymptotic waves. They will not be *plane waves*. For a finite source, the outgoing waves will be *spherical waves*.

5.

We have (this is exact)

$$\vec{A}(\vec{x}) \;=\; \int \frac{d^3x'\,\vec{J}(\vec{x}\,')}{c\,|\vec{x}-\vec{x}\,'|}\;\; e^{\mathrm{ik}\,|\vec{x}-\vec{x}'|}$$

where $\mathrm{k} \equiv \omega/\mathrm{c}$.

In the far zone,

$$|\,\vec{x}-\vec{x}\,'\,| = \mathrm{SQRT}[\,r^2 - 2\,\vec{x}\cdot\vec{x}\,' + r'^2]$$

$$\approx\; \mathrm{r} - \hat{n}\cdot\vec{x}\,' + O(\,d^2/r) \text{ where } \hat{n} = \vec{x}\,/\mathrm{r}\,.$$

Therefore (far zone approximation)

$$\vec{A}(\,\vec{x}) = \vec{A}_{\mathrm{rad}}(\,\vec{x}) + \mathrm{O}\,(1/r^2)$$

where

$$\vec{A}_{\mathrm{rad}}(\,\vec{x}) = \frac{e^{ikr}}{\mathrm{cr}}\int d^3x'\,\vec{J}(\,\vec{x}\,')\,\times$$
$$\times\,\exp\{\,-\mathrm{i}\,\mathrm{k}\,\hat{n}\bullet\vec{x}\,'\,\}$$

.5

The result:

- a spherical wave : $(1/\mathrm{r})\, e^{i(\mathrm{kr}-\omega t)}$
- with angular modulation : $\vec{\mathscr{J}}\,(\mathrm{k}\,\hat{n})$

.

Define $\vec{\mathscr{J}}(\vec{k}) = \frac{1}{c}\int d^3x'\,\vec{J}(\,\vec{x}\,')\,\times$
$$\times\,\exp\{\,-\mathrm{i}\,\vec{k}\bullet\vec{x}\,'\,\}$$

.

We have $\vec{A}_{\mathrm{rad}}(\vec{x}) = (\,e^{ikr}/\,\mathrm{r}\,)\;\vec{\mathscr{J}}(k\hat{n})\,.$

6.

The angular distribution of radiated power, in the far zone

First determine $\vec{B}_{\text{rad}}(\vec{x})$ and $\vec{E}_{\text{rad}}(\vec{x})$.

In[201]:= scanP11

Out[201]=

Start with

$$\vec{A}_{rad.}(\vec{x},t) = \frac{e^{ikr}}{cr}\int d^3x'\, \vec{J}(\vec{x}')\, e^{-ik\hat{n}\cdot\vec{x}'}$$

↳ understood where $k \equiv \omega/c$.

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{c}\int d^3x'\, \nabla\left[\frac{e^{ikr}}{r}e^{-ik\hat{n}\cdot\vec{x}'}\right]\times\vec{J}(\vec{x}')$$

$$\nabla[\ldots] = e^{ik(r-\hat{n}\cdot\vec{x}')}\left[\frac{ik}{r}\hat{n} - \frac{1}{r^2}\hat{n} - \frac{ik}{r}x_j'\,\nabla\hat{n}_j\right]$$

$$\nabla r = \hat{n}$$

$$\nabla\hat{n}_j = \nabla\left(\frac{x_j}{r}\right) = \frac{\hat{e}_j}{r} - \frac{x_j}{r^2}\hat{n}$$

In[202]:= scanP12

Out[202]=

$$\nabla[\ldots] = e^{ik(r-\hat{n}\cdot\vec{x}')}\left\{\frac{1}{r}(ik\hat{n}) + \frac{1}{r^2}\left(-\hat{n} - ik\vec{x}' + ik(\hat{n}\cdot\vec{x}')\hat{n}\right)\right\}$$

↑ $O(1)$ ↑ $O(d/\lambda)$ ↑ $O(d/\lambda)$

NEGLECT because $\frac{ik}{r} \gg \frac{1}{r^2}$

$$\nabla[\ldots] \sim e^{ik(r-\hat{n}\cdot\vec{x}')}\frac{ik}{r}\hat{n}$$

In[203]:= scanP13

Out[203]=

$$\vec{B}(\vec{x}) \sim \frac{1}{c}\int d^3x'\, e^{ik(r-\hat{n}\cdot\vec{x}')}\frac{ik}{r}\hat{n} \times \vec{J}(\vec{x}')$$

$$= ik\,\frac{e^{ikr}}{cr}\,\hat{n}\times\int d^3x'\, e^{-ik\hat{n}\cdot\vec{x}'}\,\vec{J}(\vec{x}')$$

$$= ik\,\hat{n}\times\vec{A}(\vec{x})$$

Note: $\vec{B} \sim 1/r$ in the far zone; and $\hat{n}\cdot\vec{B}_{rad} = ($

In[204]:= **scanP14**

Out[204]=

Also, by Ampère's law ($\nabla\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$) outside of the source

$$\vec{E} = \frac{+ic}{\omega}\nabla\times\vec{B} = \frac{i}{k}\nabla\times\left[ik\hat{n}\times\vec{A}(\vec{x})\right]$$

- $\frac{\partial}{\partial x_i} x_j'$ is neglected in far zone
- $\frac{\partial}{\partial x_i} A_k = \frac{\partial}{\partial x_i}\frac{e^{ikr}}{cr}\int d^3x'\, \vec{J}_k(\vec{x}')\, e^{-ik\hat{n}\cdot\vec{x}'}$

$$\sim ik\, n_i \frac{e^{ikr}}{cr}(\ldots) = ik\, n_i A_k$$

In[205]:= **scanP15**

Out[205]=

$$\vec{E} \sim \frac{i}{k}\, ik\, ik\, \hat{n}\times(\hat{n}\times\vec{A})$$

$$\vec{E} \sim \frac{i}{k}\, ik\, ik\, \hat{n}\times\left(\frac{\vec{B}}{ik}\right) = -\hat{n}\times\vec{B}$$

$$\vec{E}_{rad} = \vec{B}_{rad}\times\hat{n}$$

∴ Note that $\vec{E}_{rad}^{\,2} = \vec{B}_{rad}^{\,2}$

& $\vec{E}_{rad}$, $\vec{B}_{rad}$, and $\hat{n}$ form an orthogonal triad.

7.

Now, the angular distribution of radiated power

You'll need this for homework problem 6-5.

Theorem.

In[206]:= Show[Import["dPavg.png", "PNG"], ImageSize → 768]

In[207]:=
$$\left(\frac{dP}{d\Omega}\right)_{avg} = \frac{da}{d\Omega}\left(\frac{dP}{da}\right)_{avg} = r^2 \frac{c}{8\pi} \mathrm{Re}\left(\hat{n}\cdot(\vec{E}\times\vec{B}^*)\right)$$

Proof.

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi}\{\mathrm{Re}\,\vec{E}(\vec{x},t)\}\times\{\mathrm{Re}\,\vec{B}(\vec{x},t)\}$$

Exercise. Show that the ***time average*** of $\vec{S}(\vec{x},t)$, for harmonic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$, is

$$\vec{S}_{avg.}(\vec{x}) = \frac{c}{8\pi}\,\mathrm{Re}\,\{\,\vec{E}(\vec{x})\times\vec{B}^*(\vec{x})\,\}$$

$$(dP/d\Omega)_{avg.} = r^2\,\hat{n}\bullet\vec{S}_{avg.} \qquad Q.E.D.$$

·7

Now use $\vec{B}_{rad} = \mathrm{ik}\,\hat{n}\times\vec{A}_{rad}$ and $\vec{E}_{rad} = \vec{B}_{rad}\times\hat{n}$.

Exercise. In the far zone,

$$\frac{dP}{d\Omega} = r^2\,\frac{c}{8\pi}\,\hat{n}\bullet\mathrm{Re}\,\{\,(\vec{B}\times\hat{n})\times\vec{B}^*\,\}$$

$$= r^2\,\frac{c}{8\pi}\,|\vec{B}|^2 \quad \Longleftarrow \quad \vec{B} = \mathrm{ik}\,\hat{n}\times\vec{A}$$

$$= r^2\,\frac{c}{8\pi}\,k^2\,|\,\hat{n}\times\vec{A}\,|^2$$

$$\Uparrow\ \vec{A} = (e^{ikr}/r)\,\vec{\mathcal{J}}(k\,\hat{n})$$

In[282]:= Show[Import["eq1122.png", "PNG"], ImageSize → 768]

Out[282]=
$$\left(\frac{dP}{d\Omega}\right)_{avg} = \frac{k^2}{8\pi c}\left|\hat{n}\times\int d^3x'\,\vec{J}(\vec{x}\,')e^{-ik\hat{n}\cdot\vec{x}\,'}\right|^2, \qquad (11.22)$$