1-

CHAPTER 11
Radiation by Systems and Point Particles

Lecture #2 on Radiation

How are E.M. waves created?

$$\vec{A}(\vec{x},t) = \frac{1}{c} \int d^3x' \, \vec{J}(\vec{x}', t - |\vec{x} - \vec{x}'|/c)$$

$$\vec{A}(\vec{x},t) = \vec{A}(\vec{x})\, e^{-i\omega t}$$

$$\vec{A}_{\mathrm{rad}}(\vec{x}) = \frac{e^{ikr}}{cr} \int d^3x' \, \vec{J}(\vec{x}')\, e^{-ik\hat{n}.\vec{x}'}$$

-1

11.1 E. M. radiation by systems: the harmonic formalism

The angular distribution of Radiated Power, in the far zone

In[942]:= `eq1111`

Out[942]=

$$\left(\frac{dP}{d\Omega}\right)_{\mathrm{avg}} = \frac{k^2}{8\pi c}\left|\hat{n} \times \int d^3x' \vec{J}(\vec{x}\,')e^{-ik\hat{n}\cdot\vec{x}\,'}\right|^2, \qquad (11.22)$$

2-

EXAMPLE 1 : A CIRCULAR LOOP OF ALTERNATING CURRENT

In[943]:= figF11p2

Out[943]=

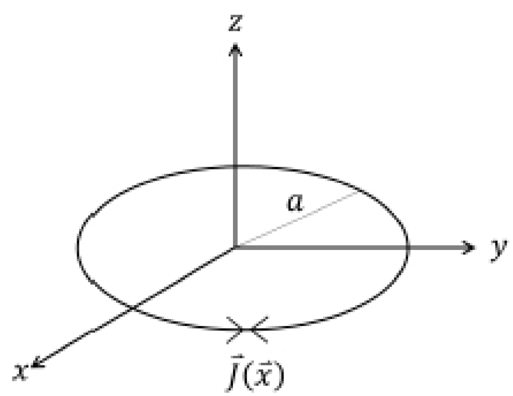


Fig. 11.2 Harmonic current loop in the *xy*-plane.

$$\vec{J}(\vec{x}\,') = \frac{I}{a}\,\delta(r' - a)\,\delta(\cos\theta')\,\hat{e}_{\phi'}$$

$$\hat{e}_{\phi'} = -\,\hat{e}_x \sin\phi' + \hat{e}_y \cos\phi'$$

$$\left(\frac{dP}{d\Omega}\right)_{avg.} = \frac{k^2 c}{8\pi}\,|\,\hat{n}\times\vec{I}(\omega)\,|^{\,2}$$

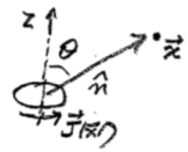
$$\vec{I}(\omega) = \frac{1}{c}\int d^3x'\,\vec{J}(\vec{x}\,')\,e^{-ik\hat{n}.\vec{x}\,'}$$

Calculation of $\vec{I}(\omega)$ and $(dP/d\Omega)_{avg.}$

In[944]:= sc1

Out[944]=

$$\vec{I}(\omega) = \frac{1}{c}\int d^3x'\,\vec{J}(\vec{x}\,')\,e^{-ik\hat{n}\cdot\vec{x}\,'}$$



$$\hat{n} = \hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta\sin\phi + \hat{e}_z\cos\theta$$

$$\vec{x}\,' = \hat{e}_x\, r'\sin\theta'\cos\phi' + \hat{e}_y\, r'\sin\theta'\sin\phi' + \hat{e}_z \cos\theta'$$

where $\theta' = \pi/2$ and $r' = a$

$$\vec{x}\,' = \hat{e}_x\, a\cos\phi' + \hat{e}_y\, a\sin\phi'$$

$$\hat{n}\cdot\vec{x}\,' = a\sin\theta\,[\cos\phi\cos\phi' + \sin\phi\sin\phi']$$

$$= a\sin\theta\,\cos(\phi' - \phi)$$

In[945]:= **sc2**

Out[945]=

$$\vec{I}(\omega) = \frac{1}{c}\frac{I}{a}\int r'^2 dr' \, d(\cos\theta') \, d\phi' \; \hat{e}_{\phi'} \times$$

$$\times\, \delta(r'-a)\,\delta(\cos\theta')\, e^{-ika\sin\theta\cos(\phi'-\phi)}$$

$$= \frac{Ia}{c}\int_0^{2\pi} d\phi' \; \hat{e}_{\phi'} \; e^{-i\Lambda\cos(\phi'-\phi)}$$

$$\Lambda \equiv ka\sin\theta$$

$$= \frac{Ia}{c}\int_0^{2\pi} d\phi' \left[-\hat{e}_x \sin\phi' + \hat{e}_y \cos\phi'\right] e^{-i\Lambda\cos(\phi'-\phi)}$$

$$= \frac{Ia}{c}\int_0^{2\pi} d\chi \left[-\hat{e}_x \sin(\phi+\chi) + \hat{e}_y \cos(\phi+\chi)\right] e^{-i\Lambda\cos\chi}$$

$$= \frac{Ia}{c}\int_0^{2\pi} d\chi \Big[\sin\chi\left(-\hat{e}_x\cos\phi - \hat{e}_y\sin\phi\right)$$
$$+ \cos\chi\left(-\hat{e}_x\sin\phi + \hat{e}_y\cos\phi\right)\Big] e^{-i\Lambda\cos\chi}$$

In[946]:= **sc3**

Out[946]=

$$= \frac{Ia}{c}\Big\{\underbrace{\left(-\hat{e}_x\sin\phi + \hat{e}_y\cos\phi\right)}_{\hat{e}_\phi}(-i)\, 2\pi J_1(\Lambda)\Big\}$$

Cylindrical Bessel function

$$\vec{I}(\omega) = \frac{-2\pi i \, Ia}{c} J_1(ka\sin\theta)\,\hat{e}_\phi$$

In[947]:= **scanEX1b**

```
Integrate[Sin[χ] * Cos[Λ * Cos[χ]], {χ, 0, 2 Pi}]
Integrate[Sin[χ] * Sin[Λ * Cos[χ]], {χ, 0, 2 Pi}]
Integrate[Cos[χ] * Cos[Λ * Cos[χ]], {χ, 0, 2 Pi}]
Integrate[Cos[χ] * Sin[Λ * Cos[χ]], {χ, 0, 2 Pi}]
```

Out[947]= 0

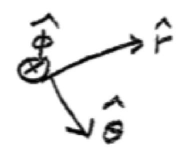
0

0

2 π BesselJ[1, Λ]

Out[933]=

$$\left(\frac{dP}{d\Omega}\right)_{avg.} = \frac{k^2 c}{8\pi}\,\Big|\underbrace{\hat{n}\times\vec{I}(\omega)}_{\propto\, \hat{r}\times\hat{\phi} \,=\, -\hat{\theta}}\Big|^2$$



$$\left(\frac{dP}{d\Omega}\right)_{avg.} = \frac{k^2 c^2}{8\pi}\left(\frac{2\pi Ia}{c}\right)^2 J_1(ka\sin\theta)^2$$

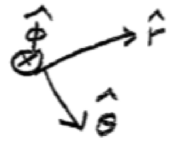
$$\left(\frac{dP}{d\Omega}\right)_{avg.} = \frac{(Iak)^2\pi}{2c} J_1(ka\sin\theta)^2$$

In[948]:= sc4

Out[948]=

$$\left(\frac{dP}{d\Omega}\right)_{avg.} = \frac{k^2 c}{8\pi} \left| \hat{n} \times \vec{I}(\omega) \right|^2$$

$$\underbrace{\phantom{\hat{n} \times \vec{I}(\omega)}}_{\propto\, \hat{r} \times \hat{\phi} \,=\, -\hat{\theta}}$$



$$\left(\frac{dP}{d\Omega}\right)_{avg.} = \frac{k^2 c}{8\pi} \left(\frac{2\pi I a}{c}\right)^2 J_1\,(ka \sin\theta)^2$$

$$\left(\frac{dP}{d\Omega}\right)_{avg.} = \frac{(I a k)^2 \pi}{2c} J_1\,(ka \sin\theta)^2$$

$$\boxed{\left(\frac{dP}{d\Omega}\right)_{avg.} = \frac{(I\,a\,k)^2\,\pi}{2\,c}\; J_1\,(\,k\,a \sin\theta\,)^2}$$

A limiting case —

For ka $\ll$ 1, approximate $J_1\,(x) \approx \frac{1}{2}\,x$;

$$\Longrightarrow \frac{dP}{d\Omega} \propto \sin^2\theta$$

In[949]:= figF11p3

Out[949]=

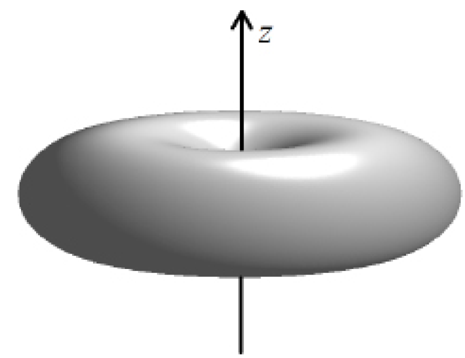
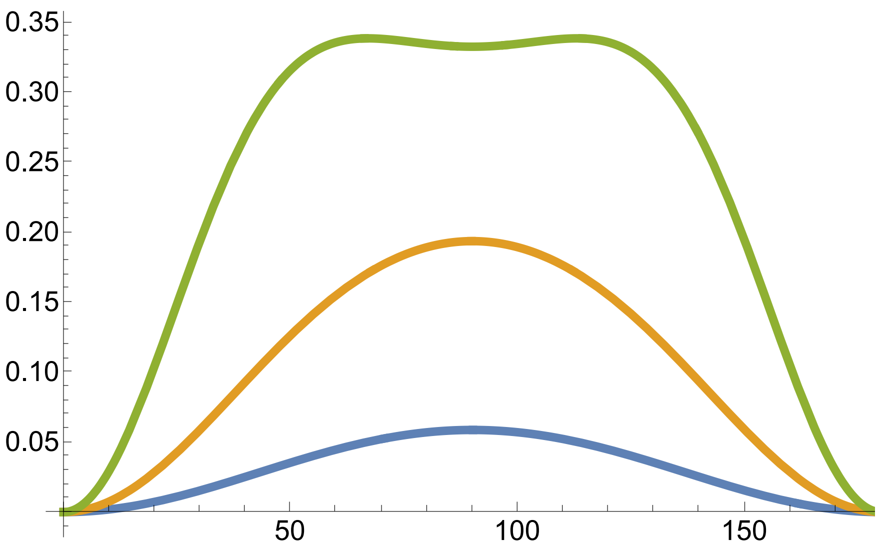


Fig. 11.3 Radiation pattern due to current loop of radius a, when $ka \ll 1$.

In[950]:=
```
Plot[{
  BesselJ[1, 0.5 * Sin[θ * Pi / 180]] ^2,
  BesselJ[1, 1.0 * Sin[θ * Pi / 180]] ^2,
  BesselJ[1, 2.0 * Sin[θ * Pi / 180]] ^2},
 {θ, 0, 180},
 PlotStyle -> Thickness[0.01], ImageSize → 768,
 BaseStyle → 24]
```

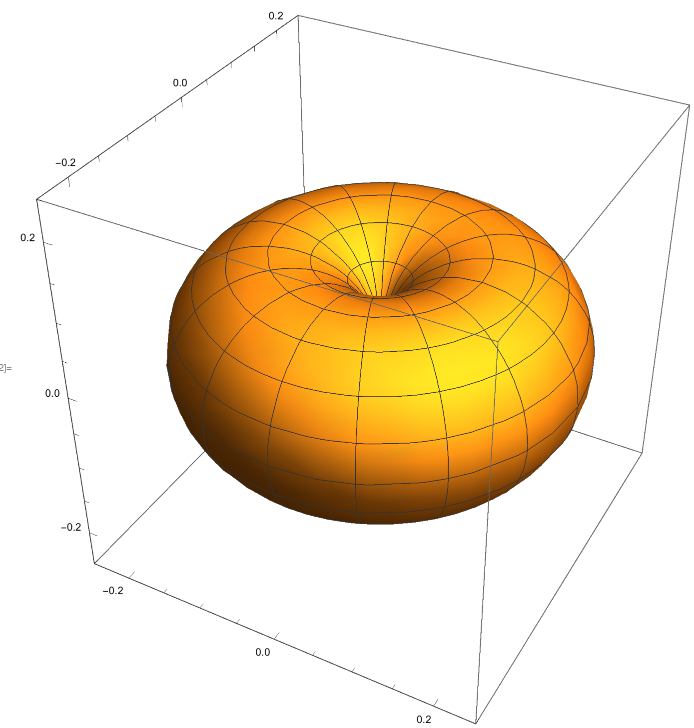
Out[950]=



In[951]:=
```
ll = 0.25
RevolutionPlot3D[
 {Sin[θ] * BesselJ[1, 0.5 * Sin[θ]],
  Cos[θ] * BesselJ[1, 0.5 * Sin[θ]]}, {θ, 0, Pi},
 PlotRange → {{-ll, ll}, {-ll, ll}, {-ll, ll}},
 ImageSize → Large]
```

Out[951]= 0.25

Out[952]=

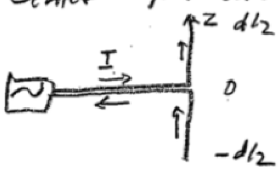


3-

Example 2 : A center - fed linear antenna

In[953]:= sc5

Center - fed linear antenna



$$J_z(\vec{x}') = I \sin\left[\frac{kd}{2} - k|z'|\right] \delta(x')\delta(y')$$

Out[953]=

$$\vec{I}(\omega) = \frac{1}{c}\int d^3x' \, \vec{J}(\vec{x}') \, e^{-ik\hat{n}\cdot\vec{x}'} \quad \hat{e}_z$$

$$= \frac{1}{c}\int_{-d/2}^{d/2} dz' \, I \sin\left[\frac{kd}{2} - k|z'|\right] e^{-ikz'\cos\theta} \quad \hat{e}_z$$

$$= \frac{I}{c}\int_{0}^{d/2} dz' \sin\left(\frac{kd}{2} - kz'\right) 2\cos[kz'\cos\theta] \; \hat{e}_z$$

$$\vec{I}(\omega) = \frac{2I}{c}\hat{e}_z \; \frac{\cos(\frac{1}{2}kd) - \cos(\frac{1}{2}kd\cos\theta)}{(-k)\sin^2\theta}$$

Mathematica!

In[954]:= sc6

Out[954]=

$$\hat{n}\times\hat{e}_z = (\hat{e}_x \sin\theta\cos\phi + \hat{e}_y \sin\theta\sin\phi)\times\hat{e}_z$$

$$= (-\hat{e}_y\cos\phi + \hat{e}_x\sin\phi)\sin\theta$$

$$= \hat{e}_\phi \sin\theta$$

$$|\hat{n}\times\vec{I}|^2 = \left(\frac{2I}{c}\right)^2 \sin^2\theta \; \frac{[\cos(\frac{1}{2}kd) - \cos(\frac{1}{2}kd\cos\theta)]^2}{k^2\sin^4\theta}$$

$$\left(\frac{dP}{d\Omega}\right)_{avg.} = \frac{I^2}{2\pi c}\, \frac{[\cos(\frac{1}{2}kd) - \cos(\frac{1}{2}kd\cos\theta)]^2}{\sin^2\theta}$$

which was used in HW problem 6-5
(Exercise 11.1.1)

In[955]:= eq1136

Out[955]=

$$\left(\frac{dP}{d\Omega}\right)_{\text{avg}} = \frac{I^2}{2\pi c}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta}\right]^2. \qquad (11.36)$$

$$\left(\frac{dP}{d\Omega}\right)_{\text{avg}} \approx \frac{I^2(kd)^4}{2\pi c}\sin^2\theta, \qquad (11.37)$$

$$kc^2 / (8\pi) \, |\hat{n}\times\vec{I}|^2$$

In[957]:= figF11p4

Out[957]=

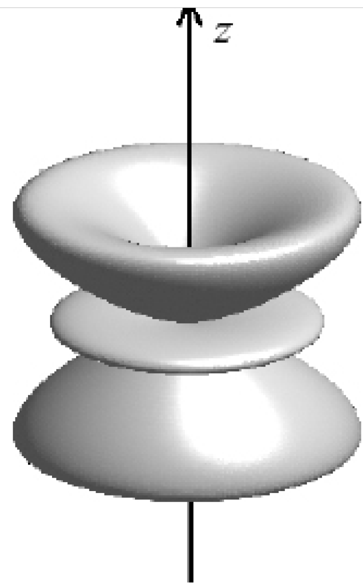


Fig. 11.4 Radiation pattern for $3.832 < ka < 7.016$.

The Mathematica command Revolution-Plot3D can make figures like this.

4-
Homework Assignment #6

Problem 6-5 :
Exercise 11.1.1 = the center-fed linear antenna

Problem 6-6 :
Exercise 11.1.3 = power radiated by an alternating current in a long straight wire

See the solutions on the course web site.