1▪

CHAPTER 11

Radiation by systems and point particles

October 8

Lecture #3 on Radiation

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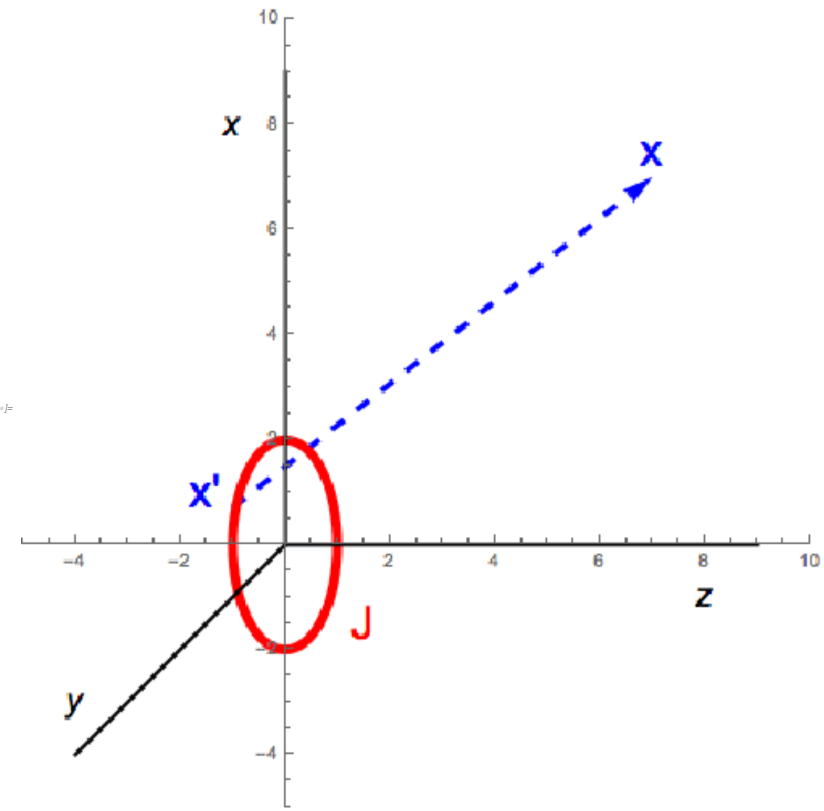
Section 11.2: "E.M. radiation by systems; the real source formalism"

Out[•]=



2▪

Electromagnetic radiation by systems; the real source formalism

The "harmonic formalism" is for harmonic systems, like antennas.

Now we'll consider arbitrary time dependence.

Start again with the potentials in the Lorenz gauge,

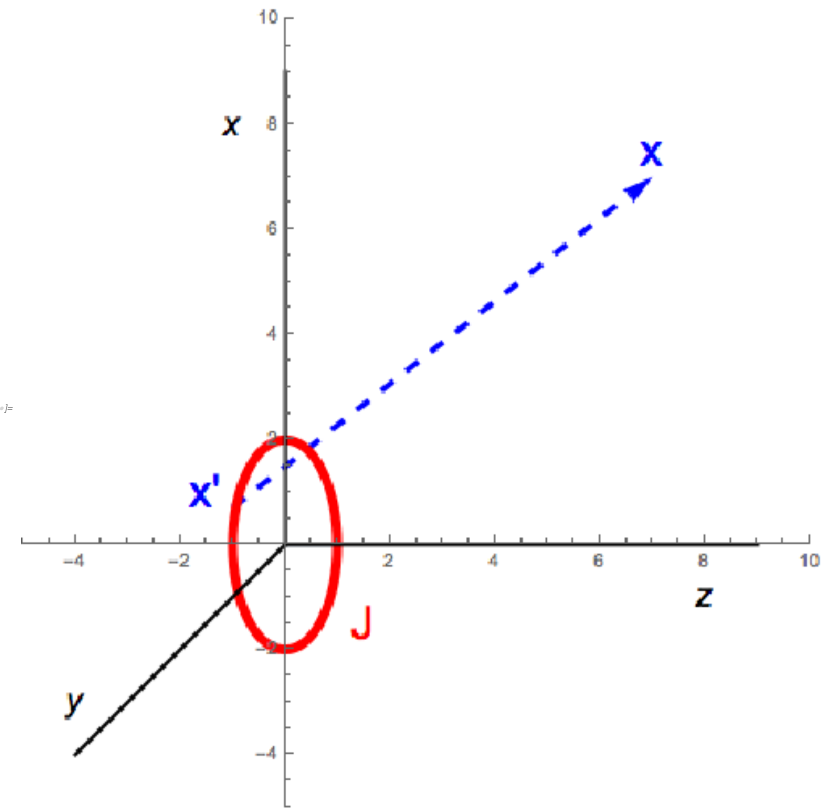
$$\Phi(\vec{x},t) = \int d^3x' \ (1/R) \ \rho(\vec{x}', t - R/c)$$

$$\vec{A}(\vec{x},t) = \int d^3x' \ 1/(cR) \ \vec{J}(\vec{x}', t - R/c)$$

where $R = |\vec{x} - \vec{x}'|$

and here $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$ are real .

Out[=]=



3▪

THE POTENTIALS IN THE FAR ZONE

For the far fields (i.e., r ≫ d ≡ size of the source) we make these approximations,

$$R = | \vec{x} - \vec{x}\,' | \approx r - \hat{n} \bullet \vec{x}\,' + O(d^2/r)$$

$$\Phi(\vec{x},t) \approx (1/r) \int d^3x' \;\; \rho(\vec{x}\,', t_r)$$

$$\vec{A}(\vec{x},t) \approx 1/(cr) \int d^3x' \;\; \vec{J}(\vec{x}\,', t_r)$$

.

The exact retarded time is

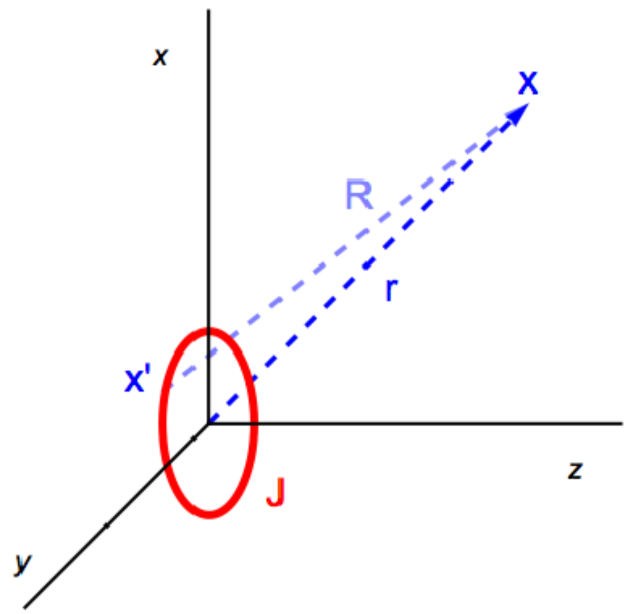
$$t_{\text{retarded}} = t - | \vec{x} - \vec{x}\,' |/c.$$

But we are making this *approximation*

$$t_r \equiv t - (r - \hat{n} \bullet \vec{x}\,')/c$$

In[•]:= R3F2

Out[•]=



4▪

THE FIELDS IN THE FAR ZONE

Now we need to calculate

$$\vec{B} = \nabla \times \vec{A} \quad \text{and} \quad \vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

Note that $\vec{A}$ and Φ depend t_r. So we'll need this chain rule:

$$\nabla f(t_r) = \nabla f(t - r/c + \hat{n} \bullet \vec{x}\,'/c)$$

$$= \frac{\partial f}{\partial t} \nabla t_r \approx \frac{\partial f}{\partial t}(-\hat{n}/c)$$

$\nabla r = \hat{n}$ (exact);

$\nabla t_r = -\hat{n}/c + O(r^{-1})$;

so the approximation is valid for large r.

I. e., we can just replace ∇ by $-\hat{n}/c\,(\partial/\partial t)$ when it acts on t_r dependence.

The magnetic field

$\vec{B} = \nabla \times \vec{A}$; calculate $\vec{B}_{\rm rad}$ $\Longrightarrow$

In[100]:= one

Out[100]=

$$A(\vec{x},t) = \frac{1}{cr}\int d^3x'\, \vec{J}(\vec{x}\,', t_r)$$

$$t_{ret} = t - |\vec{x} - \vec{x}\,'|/c$$

$$\approx t - r/c + \hat{n}\cdot\vec{x}\,'/c$$

$$\vec{B} = \nabla \times \vec{A} \approx \nabla \times \left[\frac{1}{cr}\int d^3x'\, \vec{J}(\vec{x}\,', t_r)\right]$$

$$\approx \frac{1}{cr}\nabla \times \int d^3x'\, \vec{J}(\vec{x}\,', t_r)$$

$$= \frac{1}{cr}\int d^3x'\, \frac{\partial \vec{J}}{\partial t} \times \nabla t_r \quad (-1)$$

$$\nabla t_r = -\nabla' t_r \approx -\frac{\hat{n}}{c}$$

In[101]:= two

Out[101]=

$$\vec{B} = \frac{1}{cr}\int d^3x' \, \frac{\partial \vec{J}}{\partial t} \times \frac{\hat{n}}{c} = \frac{\partial \vec{A}}{\partial t} \times \frac{\hat{n}}{c}$$

$$\vec{B} = \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \times \hat{n}$$

5 ▪ The electric field

$\vec{E} = -\nabla\Phi - \partial\vec{A}/\partial t$; calculate $\vec{E}_{\text{rad}}$

In[95]:= three

four

five

six

Out[95]=

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\nabla\Phi = \nabla \frac{1}{r}\int \rho(\vec{x}', t_r)\, d^3x' \approx \frac{1}{r}\nabla\int d^3x' \rho(\vec{x}', t_r)$$

$$= \frac{1}{r}\int d^3x' \frac{\partial \rho}{\partial t} \nabla t_r = \frac{-\hat{n}}{cr}\int d^3x' \frac{\partial \rho}{\partial t}(\vec{x}', t_r)$$

Out[96]=

Continuity equation

$$\frac{\partial \rho}{\partial t'}(\vec{x}', t') = -\nabla' \cdot \vec{J}(\vec{x}', t')$$

$$\therefore \frac{\partial \rho}{\partial t}(\vec{x}', t - \tfrac{R}{c}) = \frac{\partial \rho}{\partial t'}(\vec{x}', t') \Big]_{t' = t - R/c}$$

$$= -\nabla' \cdot \vec{J}(\vec{x}', t') \Big]_{t' = t - R/c}$$

$$= -\left\{ \nabla' \cdot \vec{J}(\vec{x}', t - \tfrac{R}{c}) - \frac{\partial \vec{J}}{\partial t} \cdot \underbrace{\nabla'[t - R/c]}_{= \hat{n}/c} \right\}$$

$$= -\nabla' \cdot \vec{J}(\vec{x}', t - R/c) + \frac{\partial \vec{J}}{\partial t} \cdot \frac{\hat{n}}{c}$$

Out[97]=

Thus, asymptotically,

$$\nabla \Phi = \frac{-\hat{n}}{cr} \int d^3x' \left\{ -\nabla' \cdot \vec{J}(\vec{x}', t - R/c) + \frac{\partial \vec{J}}{\partial t} \cdot \frac{\hat{n}}{c} \right\}$$

$$= -\hat{n} \left\{ \frac{1}{cr} \int d^3x' \, \frac{\partial \vec{J}}{\partial t} \cdot \frac{\hat{n}}{c} \right\}$$

$$= -\hat{n} \left(\frac{\partial \vec{A}}{\partial t} \right) \cdot \frac{\hat{n}}{c}$$

Out[98]=

Result, asymptotically,

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla \Phi = \frac{-1}{c}\frac{\partial \vec{A}}{\partial t} + \hat{n}\left(\frac{\partial \vec{A}}{\partial t}\right) \cdot \frac{\hat{n}}{c}$$

$$= -\frac{1}{c}\left[\frac{\partial \vec{A}}{\partial t} - \hat{n}\left(\frac{\partial \vec{A}}{\partial t} \cdot \hat{n} \right) \right]$$

$$= \frac{-1}{c} \, \hat{n} \times \left[\frac{\partial \vec{A}}{\partial t} \times \hat{n} \right]$$

$$= -\hat{n} \times \vec{B} = \vec{B} \times \hat{n}$$

6▪

Summarize the results ...

$$\vec{B}(\vec{x},t) \approx \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\times\hat{n}$$

where $\vec{A}(\vec{x},t) \approx 1/(cr)\int d^3x' \; \vec{J}(\vec{x}\,', t_r)$

and

$$\vec{E}(\vec{x},t) \approx \vec{B}(\vec{x},t)\times\hat{n}$$

Comment. Note that $\vec{E} \approx \vec{B}\times\hat{n}$ and $E^2 \approx B^2$.

(The same is true in the harmonic formalism.)

▪6

The Angular Distribution of Radiated Power in the Far Zone

Start with

$$\frac{dP}{d\Omega} = \frac{c\,r^2}{4\pi}\,\hat{n}\cdot(\vec{E}\times\vec{B})$$

$$\approx \frac{c\,r^2}{4\pi}\,\hat{n}\cdot[(\vec{B}\times\hat{n})\times\vec{B}]$$

$$\approx \frac{c\,r^2}{4\pi}\,B^2 \text{ because } \vec{B} \approx -\frac{\hat{n}}{c}\times\frac{\partial\vec{A}}{\partial t} \text{ is } \perp \hat{n}$$

$$\frac{dP}{d\Omega} = \frac{r^2}{4\pi c}\left\{\hat{n}\times\frac{\partial\vec{A}(\vec{x},t)}{\partial t}\right\}^2$$

7▪

$$\frac{dP(t)}{d\Omega} = \frac{r^2}{4\pi c} \left\{ \hat{n} \times \frac{\partial \vec{A}(\vec{x},t)}{\partial t} \right\}^2$$

(11.55)

and

$$\vec{A}(\vec{x},t) \approx 1/(cr) \int d^3x' \, \vec{J}(\vec{x}', t_r)$$

Equation (11.55) is the *instantaneous power*. How does it compare to the *time-averaged power* (11.22) in the harmonic formalism?

▪7

For example, suppose

$$\vec{J}(\vec{x}', t') = \vec{J}(\vec{x}') \cos(\omega t') .$$

$\Rightarrow \vec{A}(\vec{x},t) \approx 1/(cr) \int d^3x' \, \vec{J}(\vec{x}') \, (-\omega) \sin(\omega t_r)$

$\Rightarrow \partial\vec{A}/\partial t \approx 1/(cr) \int d^3x' \, \vec{J}(\vec{x}') \, (-\omega) \sin(\omega t_r)$

$\therefore \; dP(t)/d\Omega \approx$

$$\frac{r^2}{4\pi c} (\omega/cr)^2 \left[\int d^3x' \, \hat{n} \times \vec{J}(\vec{x}') \, \sin^2(\omega t_r) \right]^2$$

time average =

$$\frac{\omega^2}{8\pi c^3} \left[\int d^3x' \, \hat{n} \times \vec{J}(\vec{x}') \right]^2$$

8▪

Monday October 8

In[◦]:= Exercise1121

Exercise 11.2.1. You are given the time dependent sources (see Section 6.8 to see how these arise):

Out[◦]=

electric dipole at origin: $\rho(\vec{x},t) = -\vec{p}(t)\cdot\vec{\nabla}\delta(\vec{x})$,

magnetic dipole at origin: $\vec{J}(\vec{x},t) = -c\vec{m}(t)\times\vec{\nabla}\delta(\vec{x})$.

Starting with the exact formula in the real formalism, derive the results ($t_0 = t - r/c$),

In[◦]:= Exercise1122

Exercise 11.2.2. Consider a thin insulating ring of radius a and charge density $\rho(\vec{x}') = \rho_0 a\delta(r' - a)\delta(\cos\theta')\sin\varphi'$ in spherical coordinates, where φ is the usual azimuthal angle. (We may set $\rho_0 = \Lambda/a^2$, where Λ is a linear charge density.) It is spun through its symmetry axis at a constant angular velocity ω. The charge density function becomes $\rho(\vec{x}', t) = \rho_0 a\delta(r' - a)\delta(\cos\theta')\sin(\varphi' - \omega t)$.

Out[◦]=

(a) Show that its exact power distribution is

$$\frac{dP(t)}{d\Omega} = \frac{\pi\rho_0^2\omega^4 a^8}{4c^3}\left[(J_0(x)+J_2(x))^2\sin^2\gamma(t_0)\right.$$
$$\left.-4J_0(x)J_2(x)\cos^2(\phi-\omega t_0)\right],$$

($t_0 = t - r/c$) where J_0, J_2 are Bessel functions, $x \equiv ka$ sin
where $k = \omega/c$, and $\gamma(t_0)$ is defined as

$$\ddot{\vec{p}}(t_0)\cdot\hat{n} \equiv |\ddot{\vec{p}}(t_0)|\cos\gamma(t_0).$$

(b) Show that the time-averaged radiation rate is

where θ is the usual polar angle. (Notice that this result
independent of φ, as it should be.)