1.

# CHAPTER 11

## Radiation by systems and point particles

Lecture #4 on Radiation

Wednesday October 10

Section 11.3 : "Frequency distribution of radiated energy; impulsive scattering"



We have derived formulas to calculate radiated power in the far zone.

■ In the real source formalism we can calculate the instantaneous power at position $\vec{x}$,

$$dP(t)/d\Omega = r^2 / (4\pi c) [\, \hat{n} \times \dot{A}(\vec{x},t)\,]^2$$

where $\vec{A}(\vec{x},t) \approx 1/(cr) \int d^3x' \, \vec{J}(\vec{x}', t_r)$ ;

$$t_r = t - (r - \hat{n} \bullet \vec{x}')/c$$

■ In the harmonic formalism, using harmonic functions such as $\vec{J}(\vec{x},t) = \vec{J}(\vec{x})\, e^{-i\omega t}$, we can calculate the time-averaged power, $(dP / d\Omega)_{avg.}$ .

Now we want to derive the *frequency distribution* for a source with arbitrary time dependence.

2.

## 11.3. Frequency distribution of radiated energy

The starting point is equation 11.54,

In[325]:= Eq1154

Out[325]= $$=\frac{1}{4\pi c}\left[\hat{n}\times\frac{1}{c}\int d^3x'\,\frac{\partial\vec{J}(\vec{x}\,',t_r)}{\partial t}\right]^2,$$

The *integrated energy* , i.e., integrated over time, is

$$\frac{dE}{d\Omega} = \int_{-\infty}^{\infty} dt\,\frac{dP(t)}{d\Omega}$$

Now use Fourier analysis to transform from the time domain to the frequency domain.



We'll use these conventions for the Fourier integrals. Given a time dependent function f(t), define the transform to the frequency domain by

$$\tilde{f}(\omega) = 1/\sqrt{2\pi}\;\int_{-\infty}^{\infty} f(t)\;e^{i\omega t}\;dt\;;$$

then the Fourier representation for f(t) is

$$f(t) = 1/\sqrt{2\pi}\;\int_{-\infty}^{\infty} \tilde{f}(\omega)\;e^{-i\omega t}\;d\omega\;;$$

and by Parseval's theorem (Exercise 11.3.1, assigned as homework)

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} \left|\tilde{f}(\omega)\right|^2 d\omega$$

3.
Now apply Parseval's theorem to equation (11.64).

In[372]:= R411

Out[372]=

$$\frac{dE}{d\Omega} = \int_{-\infty}^{\infty} dt \, \frac{dP(t)}{d\Omega}$$

$$\frac{dP(t)}{d\Omega} = \frac{r^2}{4\pi c}\left(\hat{n} \times \dot{\vec{A}}(\vec{x},t)\right)^2$$

$$\vec{A}(\vec{x},t) = \frac{1}{cr}\int d^3x' \, \vec{J}(\vec{x}', t_r)$$

$$\frac{dE}{d\Omega} = \frac{r^2}{4\pi c}\int_{-\infty}^{\infty} dt \left[\hat{n} \times \dot{\vec{A}}(\vec{x},t)\right]^2$$



In[373]:= R412

Out[373]=

Fourier analysis $\vec{A}(\vec{x},t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \tilde{A}(\vec{x},\omega) e^{-i\omega t}$

$$\dot{\vec{A}}(\vec{x},t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} (-i\omega) \tilde{A}(\vec{x},\omega) \, e^{-i\omega t}$$

$$\frac{dE}{d\Omega} = \frac{r^2}{4\pi c}\int_{-\infty}^{\infty} d\omega \left|\hat{n} \times (-i\omega)\tilde{A}(\vec{x},\omega)\right|^2$$

Parseval's theorem

$$\frac{dE}{d\Omega} = \frac{r^2}{4\pi c}\int_{-\infty}^{\infty} d\omega \, \omega^2 \left|\hat{n} \times \tilde{A}(\vec{x},\omega)\right|^2$$

$\vec{A}(\vec{x},t)$ is real $\Rightarrow \tilde{A}(\vec{x},\omega)^* = \tilde{A}(\vec{x},-\omega)$

$$\frac{dE}{d\Omega} = \frac{r^2}{2\pi c}\int_{0}^{\infty} d\omega \, \omega^2 \left|\hat{n} \times \tilde{A}(\vec{x},\omega)\right|^2$$

4.

So now we have

$$\frac{dE}{d\Omega} = \frac{r^2}{2\pi c} \int_0^\infty \omega^2 \, d\omega \; | \hat{n} \times \tilde{\boldsymbol{A}} (\vec{x},\omega) |^2$$

(11.73)

Now, dE/dΩ is the integral (over $\omega$) of the *double differential energy distribution* $d^2E / ( d\omega \, d\Omega)$; so evidently we can identify

$$\frac{d^2 E}{d\omega \, d\Omega} = \frac{r^2 \omega^2}{2\pi c} \, | \hat{n} \times \tilde{\boldsymbol{A}} (\vec{x},\omega) |^2$$

Recall the asymptotic vector potential,

$$\vec{A}(\vec{x},t) \approx \; 1/(cr) \int d^3 x' \, \vec{J}(\vec{x}', t_r) \,;$$

and rewrite it in frequency space



$$\tilde{\boldsymbol{A}}(\vec{x},\omega) = 1/\sqrt{2\pi} \int dt \, e^{i\omega t} \; 1/(cr)$$

$$\times \int d^3 x' \, \vec{J}(\vec{x}', \; t-R/c)$$

$$= 1/\sqrt{2\pi} \int dt \, e^{i\omega t} \; 1/(cr) \int d^3 x'$$

$$\times \, 1/\sqrt{2\pi} \int \tilde{\boldsymbol{J}}(\vec{x}', \; \omega') \, e^{-i\omega' (t-R/c)} \, d\omega'$$

$$= 1/(cr) \int d^3 x' \; \tilde{\boldsymbol{J}}(\vec{x}', \; \omega) \, e^{+i\omega R/c}$$

$$= e^{ikr}/(cr) \int d^3 x' \; \tilde{\boldsymbol{J}}(\vec{x}', \; \omega) \, e^{-ik(n.x')}$$

$R = r - n.x'$

5.

$$\frac{d^2 E}{d\omega\, d\Omega} = \frac{k^2}{2\pi c} \left| \hat{n} \times \int d^3x' \, \tilde{\boldsymbol{J}}(\vec{x}\,', \omega) \, e^{-i k \hat{n} \cdot x'} \right|^2$$

Note: k = $\omega$/c

(11.75)

That is the general result. Given a current density $\vec{J}(\vec{x},t)$, we transform it to frequency space $\tilde{\boldsymbol{J}}(\vec{x},\omega)$. Then the above formula is the double differential distribution of energy as a function of frequency and solid angle. We'll need this equation in future lectures.

BTW, Wilcox points out that we could have guessed this result from equation (11.22), using the harmonic method.



.5

***Example :*** *a charged particle undergoing impulsive scattering*

First, what is *impulsive scattering*?

FIGURE 11.5.

A particle with mass m and charge e (not necessarily a proton) is moving with constant velocity $\vec{v}_1$ for t < 0. At time t = 0 it is kicked ( ≡ impulsive force). Then for t > 0 it moves with constant velocity $\vec{v}_2$.

Calculate the energy radiated, as a function of frequency and solid angle.

$\vec{J}(\vec{x}\,', t') = e\, \vec{v}_1\, \delta(\vec{x}\,' - \vec{v}_1 t')$ for t' < 0;

$\vec{J}(\vec{x}\,', t') = e\, \vec{v}_2\, \delta(\vec{x}\,' - \vec{v}_2 t')$ for t' > 0;

6.

Some pretty straightforward calculations

(page 580) give

$$\int d^3 x' \,\tilde{\boldsymbol{J}}(\vec{x}\,', \omega)\, e^{-i k \hat{n} \cdot \vec{x}'} =$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{-i e \vec{v}_1}{\omega (1-\hat{n} \cdot \vec{v}_1/c)} + \frac{i e \vec{v}_2}{\omega (1-\hat{n} \cdot \vec{v}_2/c)} \right\}$$

(11.79)

Therefore the distribution of radiated energy is

$$\frac{d^2 E}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c^3} \left| \hat{n} \times \left( \frac{\vec{v}_2}{1-\hat{n} \cdot \vec{v}_2/c} - \frac{\vec{v}_1}{1-\hat{n} \cdot \vec{v}_1/c} \right) \right|^2$$



.6

<u>The nonrelativistic limit</u>

Assume $v_1 \ll$ c and $v_2 \ll$ c.

Then

$$\frac{d^2 E}{d\omega\, d\Omega} = \frac{e^2}{4\pi^2 c^3} \,|\, \hat{n} \times (\vec{v}_2 - \vec{v}_1) \,|^2$$

For example, suppose $\vec{v}_1 = u\, e_z$ and $\vec{v}_2 = -u\, e_z$.

Then

$$\frac{d^2 E}{d\omega\, d\Omega} = \frac{e^2 u^2}{\pi^2 c^3} \sin^2\theta \qquad !$$

*What is strange about this?*

*The integral over ω is infinite.*
*In fact, dE/dω is independent of ω.*

7.

We did assume an infinite impulse — an *instantaneous* change of velocity — an idealization. So the result is unphysical.

Angular dependence :

- no radiation parallel to the z axis;
- maximum radiation in the xy plane.

Wilcox makes an interesting comment about "transition radiation", but this topic is not covered in the book. See Jackson?



7.

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Hint for Exercise 11.2.2.