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CHAPTER 11
Radiation by systems and point particles

Lecture #5 on Radiation

The classical theory of radiation

$\vec{J}(\vec{x},t)$

$\Rightarrow \vec{A}(\vec{x},t)$

$\Rightarrow$ in the far zone, $\vec{A}, \vec{B}, \vec{E}, \vec{S}$

$\Rightarrow$ $dP(t)/d\Omega$ or the time average $dP_{av}/d\Omega$

$\Rightarrow d^2E / (d\Omega\, d\omega)$

Sections 11.4 and 11.5 : Multipole expansions

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11.4 — Multipole expansion; physical interpretation

We already know $\vec{A}(\vec{x},t)$ in the far zone;

i.e., $\vec{A}(\vec{x},t)$ for $r \gg d$ and $r \gg \lambda$;

$\vec{A}(\vec{x},t) \approx \frac{1}{cr} \int d^3x' \ \vec{J}(\vec{x}', \ t_r)$

where $\quad t_r = t - (r - \hat{n} \cdot \vec{x}')/c$

.

Now, consider the limit $d \ll \lambda$.

If $d \ll \lambda$, then we have something like a point-like source;

source size $\ll$ any other parameters with dimension of length.

We can expand in powers of d/λ .

$\Uparrow$ a Taylor series expansion.

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Let $t_0 = t - r/c$ (= the "origin retarded time")

Now, $\vec{J}(\vec{x}\,', t_r)$

$$= \vec{J}(\vec{x}\,', t_0 + \hat{n} \cdot \vec{x}\,'/c)$$

$$\approx \vec{J}(\vec{x}\,', t_0) + (\hat{n} \cdot \vec{x}\,'/c)\,(\partial J/\partial t)_0 + O(d^2)$$

Plug this expansion into the equation for $\vec{A}(\vec{x},t)$

⇑ the *multipole expansion*.

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Coherency and Incoherency

Wilcox: "the multipole expansion is a perturbation approximation to deal with incoherency". See Figure 11.

In[50]:= R5scan1

Out[50]=

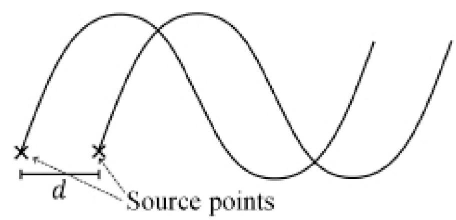

Fig. 11.6 Incoherency between radiation from different point sources, when $d \not\ll \lambda/2\pi$.

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11.5. Dipole and quadrupole contributions to radiated power

THE LOWEST ORDER TERM IN THE MULTIPOLE EXPANSION = ELECTRIC DIPOLE (E1) RADIATION

$$\vec{A}(\vec{x},t) = \frac{1}{cr} \int d^3x' \; \vec{J}(\vec{x}', t_0)$$

where $t_0 = t - r/c$.
We are assuming that the source is bounded; size ~ d.
So we can play an interesting trick.
By Gauss's theorem,

$$\int d^3x' \; \nabla' \cdot \{x'_k \vec{J}(\vec{x}', t_0)\} = 0$$

because $\vec{J} = 0$ on the surface at infinity.

Thus,

$$0 = \int d^3x' \; J_k(\vec{x}', t_0) + \int d^3x' \; x'_k \; \nabla' \cdot \vec{J}(\vec{x}', t_0)$$

By the continuity equation,

the second term

$$= -(\partial/\partial t) \int d^3x' \; x'_k \; \rho(\vec{x}', t_0)$$

$$= -\partial p_k/\partial t = -\dot{p}_k(t_0) \; ;$$

here $\vec{p}(t_0) = \int d^3x' \; \vec{x}' \; \rho(\vec{x}', t_0)$
= the dipole moment of the charge distribution, evaluated at the "origin retarded time"

Result:

$$\vec{A}(\vec{x},t) = \frac{1}{cr} \dot{\vec{p}}(t_0) \; ;$$

Equation (11.92);
vector potential of a pointlike electric dipole

The *radiated power distribution* is

$$\frac{dP(t)}{d\Omega} = \frac{r^2}{4\pi c}\left[\hat{n}\times\partial\vec{A}/\partial t\right]^2$$

$$= \frac{1}{4\pi c^3}\,[\,\hat{n}\times\ddot{\vec{p}}\,(t_0)\,]^2$$

$$= \frac{1}{4\pi c^3}\,[\,\ddot{\vec{p}}\,(t_0)\,]^2\,\sin^2\theta(t_0)$$

where $\theta(t_0)$ is the angle between $\hat{n}$ and $\ddot{\vec{p}}(t_0)$; $\sin^2\theta$ is called “dipole form” .

Total instantaneous power

Integrate over angles $\Longrightarrow$

$$P(t) = \frac{2\,[\,\ddot{\vec{p}}(t_0)\,]^2}{3\,c^3}$$

Larmor' s formula

Consider a point particle, moving non-relativistically ($v \ll c$), with charge e and position $\vec{r}$(t). The dipole moment with respect to the origin at retarded time t_0 is $\vec{p}(t_0)$ = e $\vec{r}(t_0)$.

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c^3}\,[\,\hat{n}\times\ddot{\vec{r}}\,(t_0)\,]^2$$

\acceleration\

$$P(t) = \frac{2\,e^2\,\ddot{\vec{r}}(t_0)^2}{3\,c^3} = \frac{2\,e^2\,\vec{a}(t_0)^2}{3\,c^3}$$

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THE N.L.O. TERM IN THE MULTIPOLE EXPANSION

Next, calculate the ***correction*** to electric dipole (E1) radiation.

$$\vec{A}(\vec{x},t) \approx \frac{1}{cr} \int d^3x' \, \vec{J}\left(\vec{x}',\ t_0 + \hat{n} \cdot \vec{x}'/c\right)$$

$$\approx \frac{1}{cr} \int d^3x' \left\{ \vec{J}(t_0) + \hat{n} \cdot \vec{x}'/c \ \dot{\vec{J}}(t_0) \right\}$$

(LO↗) (NLO↗)

Pages 584 - 587: The NLO term contributes *both* magnetic dipole (M1) *and* electric quadrupole (E2) contributions.

After some long calculations (3 pages) WT derive these contributions,

$$\left(\frac{dP(t)}{d\Omega}\right)_{M1} = \frac{1}{4\pi c^3} \left[\hat{n} \times \ddot{\vec{m}}(t_0) \right]^2$$

$$\left(\frac{dP(t)}{d\Omega}\right)_{E2} = \frac{1}{144\pi c^5} \left[\hat{n} \times \dddot{\vec{Q}}(\hat{n}, t_0) \right]^2$$

Eqs. 114 and 118

Limitations of these results

- We have neglected interference between radiation from $\ddot{p}$, $\ddot{m}$ and $\dddot{Q}$.
- WT give some more details on E2 radiation.
- We have only considered E1 , M1 and E2.

In[33]:= R5scan3

Out[33]=

The results in this section are limited to the lowest three moments $\vec{p}$, $\vec{m}$. and Q_{ij} from a dynamic charge distribution. For more complicated sources, many multipoles may be necessary. The situation is similar to the electrostatic case in Sections 5.1 and 5.2, where the potential, or equivalently the charge distribution, was expanded in the spherical harmonics Y_{lm}. A similar spherical multipole formalism exists for harmonic radiation based upon the vector spherical harmonics,

These advanced topics are for the experts.

One other comment:

You probably studied E1, M1 and E2 radiation from an atom or a nucleus in nonrelativistic quantum mechanics.

6■ Homework Assignment 8

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Reading: Chapter 11 Sections 6-9

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Problems: A bunch of examples

Exercise 11.5.5

Exercise 11.5.6

Exercise 11.5.7

Exercise 11.5.8

Exercise 11.6.1

Exercise 11.6.2