1-

Radiation by a point charge (WT Section 11.6, but I'm following Jackson here.)

The results are relativistically correct, although not covariant. At the end a non-relativistic limit will be written down.

Statement of the problem

We have a point particle with charge e, moving in three dimensions with trajectory $\vec{r}(t)$. The velocity of the particle is $\vec{v}(t) = d\,\vec{r}/dt$.

(a) Calculate the potentials.

(b) Calculate the fields.

(c) Calculate the radiated power.

Charge density and Current density

$\rho(\vec{x}\,',t') = e\,\delta^3[\vec{x}\,' - \vec{r}(t')]$

$J(\vec{x}\,',t') = e\,\vec{v}(t')\,\delta^3\left[\vec{x}\,' - \vec{r}(t')\right]$

The retarded potentials

Recall the retarded potentials in the Lorenz gauge,

$$\Phi(\vec{x},t) = \int d^3x'\, dt'\, \rho(\vec{x}\,',t')\, \frac{\delta(t-t'-|\vec{x}-\vec{x}\,'|/c)}{|\vec{x}-\vec{x}\,'|}$$

$$= \int d^3x'\, \frac{\rho(\vec{x}\,',t-|\vec{x}-\vec{x}\,'|/c)}{|\vec{x}-\vec{x}\,'|}$$

$$A(\vec{x},t) = \int d^3x'\, dt'\, J(\vec{x}\,',t')\, \frac{\delta(t-t'-|\vec{x}-\vec{x}\,'|/c)}{c|\vec{x}-\vec{x}\,'|}$$

$$= \int d^3x'\, \frac{\vec{J}(\vec{x}\,', t-|\vec{x}-\vec{x}\,'|/c)}{c|\vec{x}-\vec{x}\,'|}$$

2-

The Liénard-Wiechert potentials

See WT Exercise 8.?.?.

Use $\delta^3[\vec{x}' - \vec{r}(t')]$ to do the integral over $\vec{x}'$.

$$\Phi(\vec{x},t) = \int d^3x' \frac{e\ \delta^3[\vec{x}' - \vec{r}(t - |\vec{x} - \vec{x}'|/c)]}{|\vec{x} - \vec{x}'|}$$

Define

$$t_{ret} = t - |\vec{x} - \vec{r}(t_{ret})|/c$$

Note that t_{ret} is a function of $\vec{x}$ and t .

We do not have an explicit equation for $t_{ret}(\vec{x},t)$. But we do have an *implicit* equation.

The delta function $\delta^3[\vec{x}' - \vec{r}(t - |\vec{x} - \vec{x}'|/c)]$ is a sharply peaked distribution, sharply peaked at $\vec{x}' = \vec{r}(t_{ret})$.

Don't forget,

$$\int_{-\infty}^{\infty} du\ \ \delta[f(u)]\ g(u) = \frac{g(u_0)}{|f'(u_0)|}$$

where $f(u_0) = 0$.

Or,

$$\int_{-\infty}^{\infty} d^3u\ \ \delta^3[\vec{f}(\vec{u})]\ g(\vec{u}) = \frac{g(\vec{u}_0)}{|\mathrm{DET}\ \partial f_i/\partial u_j|_0}$$

3-

$$\Phi(\vec{x},t) = \int d^3x' \frac{e\ \delta^3[\ \vec{x}' - \vec{r}(t - |\vec{x} - \vec{x}'|/c)\ ]}{|\ \vec{x} - \vec{x}'\ |}$$

$$= \left\{ \frac{e}{|\vec{x} - \vec{x}'\ |} \blacksquare \frac{1}{\mathrm{DET}\left\{ \delta_{ij} - \left(\dot{r}_i/c\right)(x - x')_j / |\vec{x} - \vec{x}'| \right\}} \right\}_{x' = r(t_{ret})}$$

$$= \left\{ \frac{e}{|\ \vec{x} - \vec{r}| - (\ \vec{v}/c\ ) \blacksquare (\ \vec{x} - \vec{r}\ )} \right\}_{ret}$$

□

$$\vec{A}(\vec{x},t) = \int d^3x' \frac{e\,\vec{v}(t')\,\delta^3[\ \vec{x}' - \vec{r}(t - |\vec{x} - \vec{x}'|/c)\ ]}{c|\ \vec{x} - \vec{x}'\ |}$$

$$= \left\{ \frac{e\,\vec{v}/c}{|\vec{x} - \vec{r}\ | - (\ \vec{v}/c\ ) \blacksquare (\ \vec{x} - \vec{r}\ )} \right\}_{ret}$$

These are the Liénard-Wiechert potentials.

Notation: $[f(t')]_{ret} \equiv f(t_{ret})$

The fields, $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \text{ and } \vec{B} = \nabla \times \vec{A}\,.$$

This will be difficult, because we'll need to use the chain rule for $t_{ret}(\vec{x}, t)$.

$$t_{ret} = t - (1/c)\ |\ \vec{x} - \vec{r}(\ t_{ret}\ )\ |$$

Keeping $\vec{x}$ constant,

$$dt_{ret} = dt + \{\ (\ \vec{v}/c) \bullet (\vec{x} - \vec{r}) / |\vec{x} - r|\ \}_{ret}\ dt_{ret}$$

$dR = \vec{R} \cdot d\vec{R}/R$

Notation: $\vec{\beta} = \vec{v}/c$

$$\therefore \frac{\partial t_{ret}}{\partial t} = \left(\frac{1}{1 - \vec{\beta} \blacksquare \vec{R}/R} \right)_{ret}$$

Notation : $\vec{R} = \vec{x} - \vec{r}$

4-

Calculate $\vec{E}(\vec{x},t)$

$\vec{E} = -\nabla\Phi - (1/c)\, \partial \vec{A}/\partial t$

Requires some cleverness.

Takes about 5 pages of algebra.

See Jackson.

Calculate $\vec{B}(\vec{x},t)$

$\vec{B} = \nabla \times \vec{A}$

The calculation is similar to $\vec{E}$.

The result is remarkable simple.

See Jackson.

$\vec{B} = [\, \hat{R} \times \vec{E}\,]_{ret}$

5-
Results: Jackson Equations (14.14) and (14.13)

In[=]:= J1414

Out[=]=
$$\boldsymbol{E} = q\left[\frac{(1-\beta^2)(\hat{\boldsymbol{R}}-\boldsymbol{\beta})}{R^2(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^3}\right]_{\text{ret}} + \frac{q}{c}\left[\frac{\hat{\boldsymbol{R}}\times[(\hat{\boldsymbol{R}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{R(1-\boldsymbol{\beta}\cdot\hat{\boldsymbol{R}})^3}\right]_{\text{ret}}$$

and

$$\vec{B} = [\ \hat{R} \times \vec{E}\ ]_{\text{ret}}$$

.

Quick check: What are $\vec{E}$ and $\vec{B}$ if $\vec{v}$ = 0?

Not so quick check: What are $\vec{E}$ and $\vec{B}$ if $\vec{v}$ is constant?

The Larmor formula

Now we are ready to calculate the power radiated by the particle.

▫

The energy passing through solid angle dΩ during time interval dt is

$$dE = \frac{dP(t)}{d\Omega}\, d\Omega\, dt$$

$$= \lim_{R\to\infty} R^2 \frac{c}{4\pi} \hat{R} \bullet (\vec{E}\times\vec{B})\, d\Omega\, dt\,.$$

6-

For the limit we only need the asymptotic fields, i.e., terms proportional to 1/R;

we can neglect the "velocity term" in $\vec{E}(\vec{x},\mathrm{t})$ because that goes like $1/R^2$.

In the limit, $\vec{E}$, $\vec{B}$, and $\hat{R}$ form an orthogonal triad.

$$dE = \lim R^2 \ \frac{c}{4\pi} \ \hat{R}\bullet[\vec{E}\times(\hat{R}\times\vec{E})]\ d\Omega\ dt$$

$$= \lim R^2 \ \frac{c}{4\pi} \ E^2\ d\Omega\ dt$$

$$= \frac{e^2}{4\pi c} \ \frac{\left\{\hat{R}\times\left[\left(\hat{R}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right\}^2}{\left(1-\beta\bullet\hat{R}\right)^6} \ d\Omega\ dt$$

Evaluation at t_{ret} is understood.
I.e., RHS means $[RHS]_{ret}$

Now, what do we want ?

Should we calculate the power in “far observer time” or in “particle time”?

Usually the more interesting is the amount of energy radiated per unit of *particle* time.

I.e., $dP(t_{\mathrm{ret}})/d\Omega$

$$= \frac{e^2}{4\pi c} \ \frac{\left\{\hat{R}\times\left[\left(\hat{R}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right]\right\}^2}{\left(1-\vec{\beta}\bullet\hat{R}\right)^5}$$

because $dt/dt_{\mathrm{ret}} = 1-\vec{\beta}\bullet\hat{R}$.

Evaluation at t_{ret} is understood.
I.e., RHS means $[RHS]_{ret}$

7-

Angular dependence and total power

First, expand the double cross product *squared;*

$$\left\{ \hat{R} \times \left[\left(\hat{R} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right] \right\}^2 .$$

Then integrate over the solid angle to get the total power.

WLOG: Let the z-axis be the direction of $\vec{\beta}$ (i.e., the direction of velocity at the time t_r), and integrate over θ and $\phi \equiv$ the polar angles of $\hat{R}$. This will involve quite a bit of algebra and calculus, so let's do it by Mathematica.

Mathematica Calculations

```
In[◦]:= Rh = {Sin[θ] Cos[ϕ], Sin[θ] Sin[ϕ], Cos[θ]};
β = {0, 0, b};
βd = {αx, 0, αz};
(* wlog let βd be in the xz plane *)
V1 = Cross[(Rh - β), βd];
V2 = Cross[Rh, V1];
num = Dot[V2, V2] // Expand // FullSimplify
```

Out[◦]=

$$\frac{1}{4}\left(3\left(1+b^2\right)\alpha x^2+2\,\alpha z^2+\left(\left(1+b^2\right)\alpha x^2-2\,\alpha z^2\right)\mathrm{Cos}[2\,\theta]-8\,\alpha x\,\mathrm{Cos}[\theta]\left(b\,\alpha x+\alpha z\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]\right)+2\,\alpha x\,\mathrm{Sin}[\theta]\left(4\,b\,\alpha z\,\mathrm{Cos}[\phi]+\left(-1+b^2\right)\alpha x\,\mathrm{Cos}[2\,\phi]\,\mathrm{Sin}[\theta]\right)\right)$$

```
In[◦]:= A1 = Integrate[num, {ϕ, 0, 2 Pi}]
```

Out[◦]=

$$\frac{1}{2}\pi\left(3\left(1+b^2\right)\alpha x^2+2\,\alpha z^2-8\,b\,\alpha x^2\,\mathrm{Cos}[\theta]+\left(\left(1+b^2\right)\alpha x^2-2\,\alpha z^2\right)\mathrm{Cos}[2\,\theta]\right)$$

In[∘]:=
```
A2 = A1 // TrigExpand;
A2 = A2 /. {Cos[θ] → u, Sin[θ] → Sqrt[1 - u^2]}
A2 = A2 // Simplify
```

Out[∘]=
$$\frac{3\,\pi\,\alpha x^2}{2} + \frac{3}{2}\,b^2\,\pi\,\alpha x^2 - 4\,b\,\pi\,u\,\alpha x^2 + \frac{1}{2}\,\pi\,u^2\,\alpha x^2 + \frac{1}{2}\,b^2\,\pi\,u^2\,\alpha x^2 - \frac{1}{2}\,\pi\,\left(1 - u^2\right)\,\alpha x^2 - \frac{1}{2}\,b^2\,\pi\,\left(1 - u^2\right)\,\alpha x^2 + \pi\,\alpha z^2 - \pi\,u^2\,\alpha z^2 + \pi\,\left(1 - u^2\right)\,\alpha z^2$$

Out[∘]=
$$\pi\,\left(\left(1 - 4\,b\,u + u^2 + b^2\,\left(1 + u^2\right)\right)\,\alpha x^2 - 2\,\left(-1 + u^2\right)\,\alpha z^2\right)$$

In[∘]:=
```
A3 = Assuming[b < 1 && b > 0,
  Integrate[A2 / (1 - b * u) ^5, {u, -1, 1}]]
```

Out[∘]=
$$\frac{8\,\pi\,\left(\left(-1 + b^2\right)\,\alpha x^2 - \alpha z^2\right)}{3\,\left(-1 + b^2\right)^3}$$

In[∘]:=
```
(* Compare *)
β
βd
μ = Cross[β, βd]
Dot[βd, βd] - Dot[μ, μ] // Simplify
```

Out[∘]= $\{0, 0, b\}$

Out[∘]= $\{\alpha x, 0, \alpha z\}$

Out[∘]= $\{0, b\,\alpha x, 0\}$

Out[∘]= $-\left(-1 + b^2\right)\,\alpha x^2 + \alpha z^2$

8-

Total power w.r.t. particle time ; Jackson Eq. (14.26)

$$P = \frac{2e^2\left[\left(\dot{\beta}\right)^2 - \left(\beta \times \dot{\beta}\right)^2\right]}{3c(1-\beta^2)^3}$$

The nonrelativistic limit

$\beta = \mathrm{v}/\mathrm{c} \ll 1$

$\Longrightarrow$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}(\hat{R} \times \dot{\boldsymbol{v}})^2$$

$$P = \frac{2e^2}{3c^3}(\dot{\boldsymbol{v}})^2$$

the Larmor formula

▫

In classical electrodynamics, when an isolated charge undergoes acceleration it must radiate.