Radiation by a point particle with a specified trajectory

Given charge e with trajectory $\vec{r}(t)$

$$\rho(\vec{x},t) = e\,\delta^3(\vec{x}-\vec{r}(t)) \quad \text{and} \quad \vec{J}(\vec{x},t) = \vec{v}(t)\,\rho(\vec{x},t)$$

The Liénard Wiechert potentials (Exercise 8.6.7)

$$\Phi(\vec{x},t) = \left[\frac{e}{R - \vec{\beta}\cdot\hat{R}}\right]_{ret}$$

$$\vec{A}(\vec{x},t) = [\vec{\beta}]_{ret}\,\Phi(\vec{x},t)$$

<u>Notation</u> $\quad [Q(t)]_{ret} = Q(t_{ret})$

where $\quad t_{ret} = t - \frac{1}{c}|\vec{x} - \vec{r}(t_{ret})|$ (retarded time)

Also, $\vec{R} = \vec{x} - \vec{r}(t)$, $R = |\vec{R}|$, $\hat{R} = \dfrac{\vec{R}}{R}$

Also, $\vec{\beta}(t) = \dfrac{\vec{v}(t)}{c}$ and $\vec{v} = \dot{\vec{r}}$.

The <u>complete fields</u> of the point charge

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla\times\vec{A}.$$

From the previous lecture,

$$\vec{E}(\vec{x},t) = \left[\frac{e(\hat{R}-\vec{\beta})(1-\beta^2)}{R^2(1-\vec{\beta}\cdot\hat{R})^3}\right]_{ret} + \left[\frac{e\hat{R}\times[(\hat{R}\times\vec{\beta})\times\vec{a}]}{c^2R(1-\vec{\beta}\cdot\hat{R})^3}\right]_{ret}$$

$$\vec{a} = \dot{\vec{v}}$$

$$\vec{E} = \vec{E}_{vel.} + \vec{E}_{accel.}$$

$$t_{ret} = t - |\vec{x} - \vec{r}(t_{ret})|/c$$

$$\vec{B}(\vec{x},t) = [\hat{R}]_{ret} \times \vec{E}(\vec{x},t)$$

<u>Radiated Power</u> per unit solid angle,

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c}\left[\frac{\{\hat{R}\times[(\hat{R}-\vec{\beta})\times\dot{\vec{\beta}}]\}^2}{(1-\vec{\beta}\cdot\hat{R})^6}\right]_{ret.}$$

Here

t = observation time at far distance.

This has a limit $R\to\infty$ implied, so

$$\vec{R} = \vec{x} - \vec{r}(t_{ret}) \simeq \vec{x}$$

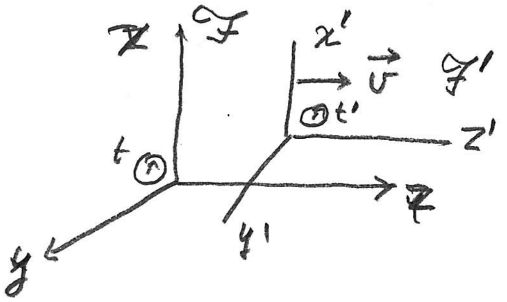
$$R = |\vec{x}-\vec{r}| \simeq |\vec{x}| - \hat{n}\cdot\vec{r}$$

$$\hat{R} \simeq \hat{n} = \vec{x}/|\vec{x}|$$

Example $\vec{a} = 0$

Derivation of the Lienard Wiechert potentials, from a Lorentz boost from the rest frame of e

The charge has constant velocity, $\vec{v}$, so there is no radiation.



Recall the Lorentz transformations

$$ct' = \gamma(ct - \beta x_{\parallel})$$
$$x'_{\parallel} = \gamma(x_{\parallel} - \beta ct)$$
$$\vec{x}'_{\perp} = \vec{x}_{\perp}$$

or

$$ct = \gamma(ct' + \beta x'_{\parallel})$$
$$x'_{\parallel} = \gamma(x_{\parallel} + \beta ct')$$
$$\vec{x}'_{\perp} = \vec{x}_{\perp}$$

Also,

$$\Phi = \gamma(\Phi' + \beta A'_{\parallel})$$
$$A_{\parallel} = \gamma(A'_{\parallel} + \beta \Phi')$$
$$\vec{A}_{\perp} = \vec{A}'_{\perp}$$

$\mathcal{F}'$ is the rest frame of e.
(e is at the origin of $\mathcal{F}'$, moving with velocity $\vec{v}$ in frame $\mathcal{F}$.)
The potentials in frame $\mathcal{F}'$ are

$$\Phi'(\vec{x}') = \frac{e}{|\vec{x}'|} \quad \text{and} \quad \vec{A}' = 0.$$

Now obtain $\Phi(\vec{x}t)$ and $\vec{A}(\vec{x}t)$ by L. transformation.

$$\Phi(\vec{x},t) = \gamma\, \Phi'(\vec{x}',t') = \frac{\gamma e}{\sqrt{(x_\parallel')^2 + (\vec{x}_\perp')^2}}$$

$$= \frac{\gamma e}{\sqrt{\gamma^2 (x_\parallel - \beta c t)^2 + \vec{x}_\perp^{\,2}}}$$

$\vec{r}(t) = \vec{v}\, t$

Now recall $t_{ret} = t - \frac{1}{c} |\vec{x} - \vec{v}\, t_{ret}|$

$$t_{ret} = t - \frac{1}{c}\sqrt{(x_\parallel - v t_{ret})^2 + \vec{x}_\perp^{\,2}}$$

Let $\xi = x_\parallel - v t_{ret}$.

Then $t_{ret} = t - \frac{1}{c}\sqrt{\xi^2 + x_\perp^2}$

$$t - t_{ret} = \frac{1}{c}\sqrt{\xi^2 + x_\perp^2}$$

Now

$$\Phi(\vec{x},t) = \gamma e \left\{ \gamma^2 (x_{\parallel} - v t_{ret} + v t_{ret} - v t)^2 + x_{\perp}^2 \right\}^{-1/2}$$

$$= \gamma e \left\{ \gamma^2 \left(\xi - \frac{v}{c} \sqrt{\xi^2 + x_{\perp}^2} \right)^2 + x_{\perp}^2 \right\}^{-1/2}$$

$$\frac{1}{\gamma^2} = 1 - \beta^2$$

$$= e \left\{ (\xi - \beta \sqrt{\xi^2 + x_{\perp}^2})^2 + x_{\perp}^2 (1 - \beta^2) \right\}^{-1/2}$$

$$= e \left\{ \xi^2 + \beta^2 (\xi^2 + x_{\perp}^2) - 2 \xi \beta \sqrt{\xi^2 + x_{\perp}^2} + x_{\perp}^2 (1 - \beta^2) \right\}^{-1/2}$$

$$= e \left\{ \xi^2 (1 + \beta^2) + x_{\perp}^2 - 2 \xi \beta \sqrt{\xi^2 + x_{\perp}^2} \right\}^{-1/2}$$

$$= e \left\{ (\sqrt{\xi^2 + x_{\perp}^2} - \xi \beta)^2 \right\}^{-1/2}$$

$$= \frac{e}{\sqrt{\xi^2 + x_{\perp}^2} - \xi \beta}$$

Remember : $\vec{R}(t_{ret}) = \vec{x} - \vec{v}\, t_{ret}$

$$R = \sqrt{(x_{\parallel} - v t_{ret})^2 + x_{\perp}^2} = \sqrt{\xi^2 + x_{\perp}^2}$$

$$\vec{\beta} \cdot \vec{R} = \beta x_{\parallel} - \beta v t_{ret} = \beta \xi$$

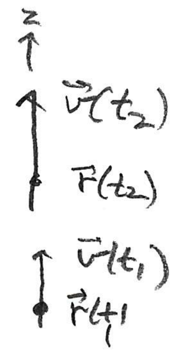
$$\Phi(\vec{x},t) = \left[\frac{e}{R - \vec{\beta} \cdot \vec{R}} \right]_{ret}$$ ← Liénard - Wiechert

Radiated power per unit solid angle

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c}\left[\frac{\{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]\}^2}{(1-\vec{\beta}\cdot\hat{n})^6}\right]_{ret}$$

(far zone, approximating $\hat{R}\approx\hat{n}$) (11-146)

<u>Example 1</u> Calculate the power distribution where $\dot{\vec{v}}$ is parallel to $\vec{v}$, as in a <u>linear</u> <u>accelerator</u>.



Let $\vec{v}$ be the z direction,

$$\vec{v} = v\hat{e}_z = c\beta\,\hat{e}_z$$
$$\dot{\vec{v}} = \dot{v}\,\hat{e}_z = c\dot{\beta}\,\hat{e}_z$$

$$\{\ldots\} = \hat{n}\times\left[(\hat{n}-\beta\hat{e}_z)\times\dot{\beta}\,\hat{e}_z\right]$$
$$= \hat{n}\times\left[\hat{n}\times\hat{e}_z\right]\dot{\beta}$$
$$= \left[\hat{n}\cos\theta - \hat{e}_z\right]\dot{\beta}$$

$$\{\ldots\}^2 = (\cos^2\theta + 1 - 2\cos^2\theta)\,\dot{\beta}^2 = \dot{\beta}^2\sin^2\theta$$

and $1-\vec{\beta}\cdot\hat{n} = 1-\beta\cos\theta$

$$\frac{dP(t)}{d\Omega} = \frac{e^2 \dot{\beta}^2}{4\pi c} \frac{\sin^2\theta}{(1-\beta\cos\theta)^6}$$

$\frac{dP(t)}{d\Omega}$ is maximum in the direction θ_{max}

$$\cos\theta_{max} = \frac{1}{4\beta}\left\{ \sqrt{1+24\beta^2} - 1 \right\}$$

The total emitted power

$$P(t_r) = \int d\Omega \frac{dP(t)}{d\Omega} \underbrace{\frac{dt}{dt_{ret}}}_{\to [1-\hat{n}\cdot\vec{\beta}]_{ret}}$$

$$P(t_r) = \frac{2e^2}{3c} \frac{\dot{\beta}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2}{(1-\beta^2)^3} \quad \text{"the Liénard result"}$$

$$= \frac{2e^2}{3c}\frac{\dot{\beta}^2}{(1-\beta^2)^3} = \frac{2e^2}{3c}\dot{\beta}^2\gamma^6$$

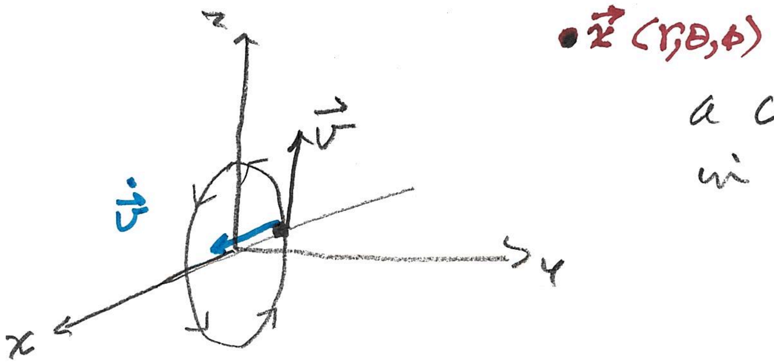
Nonrelativistic Limit

$$P(t_r) = \frac{2e^2}{3c^2}\dot{v}^2 \quad \text{(Larmor's formula)}$$

Example 2

Calculate the power distribution when $\dot{\vec{v}}$ is perpendicular to $\vec{v}$, as in a circular accelerator

Again, let $\vec{\beta}$ be in the z direction, $\vec{\beta} = \beta \hat{e}_z$; and let $\dot{\vec{\beta}}$ be in the x direction, $\dot{\vec{\beta}} = \dot{\beta} \hat{e}_x$.



a circular orbit in the xz plane.

Now $\quad \hat{n} = \hat{e}_x \sin\theta \cos\phi + \hat{e}_y \sin\theta \sin\phi + \hat{e}_z \cos\theta$

$$\left\{ \hat{n} \times \left[(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}} \right] \right\} = \dot{\beta} \left\{ \hat{n} \times \left[(\hat{n} - \beta \hat{e}_z) \times \hat{e}_x \right] \right\}$$

$$= \dot{\beta} \left\{ \hat{n} \times \left[-\hat{e}_z \sin\theta \sin\phi - \beta \hat{e}_y \right] \right\}$$

$$= \dot{\beta} \left\{ -\sin\theta \sin\phi \left(-\hat{e}_y \sin\theta \cos\phi + \hat{e}_x \sin\theta \sin\phi \right) - \beta \left(\hat{e}_z \sin\theta \cos\phi - \hat{e}_x \cos\theta \right) \right\}$$

$\{ \dots \}^2$ = a lot of algebra

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c^3} \frac{\dot{\beta}^2}{(1-\beta\cos\theta)^4}\left[1 - \frac{(1-\beta^2)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^2}\right]$$

$$P(t_r) = \frac{2e^2}{3c}\frac{\dot{\beta}^2}{(1-\beta^2)^2} = \frac{2e^2}{3c}\dot{\beta}^2\gamma^4$$