Section 11.8
Radiation by a particle in periodic motion

Review: Energy radiated by a moving charge

Using the real source formalism ...

Let $\vec{r}(t)$ be the particle trajectory in three dimensions.

We already know

In[1355]:= scan81

Out[1355]=

$$\frac{d^2E}{d\omega\, d\Omega} = \frac{\omega^2 r^2}{2\pi c} \left| \hat{n} \times \vec{A}(\vec{x},\omega) \right|^2 \qquad (74) \qquad (211)$$

$$\vec{A}(\vec{x},\omega) = \frac{1}{cr} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}}\, e^{i\omega t} \int d^3x' \, \vec{J}(\vec{x}', t_r)$$

$$= \frac{e^{ikr}}{cr} \int d^3x' \, \vec{J}(\vec{x}', \omega)\, e^{-ik\hat{n}\cdot\vec{x}'} \qquad (71)\ (212)$$

$$\vec{J}(\vec{x},t) = e\,\vec{v}(t)\,\delta^3(\vec{x} - \vec{r}(t))$$

$$\vec{A}(\vec{x},\omega) = \frac{e^{ikr}}{cr} \frac{e}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} dt\, \vec{v}(t)\, e^{i\omega(t - \frac{1}{c}\hat{n}\cdot\vec{r}(t))}}_{\equiv\, \vec{\chi}}$$

$$\therefore \frac{d^2E}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c^3} \left| \hat{n} \times \vec{\chi} \right|^2$$

where $\vec{\chi}$ is defined by

$$\vec{\chi}(\hat{n},\omega) \equiv \int_{-\infty}^{\infty} dt\, \vec{v}(t)\; e^{i\omega t}\, e^{-i(\omega/c)\,\hat{n}\cdot\vec{r}(t)}$$

$$\left| \vec{x} - \vec{r}(t) \right| \approx |\vec{x}| - \hat{n} \cdot \vec{r}(t)$$

Then

$$\frac{d^2E}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2 c^3} \left| \hat{n} \times \vec{\chi} \right|^2$$

Periodic motion

Actually, the motion cannot extend over infinite time. We'll divide the time axis into three parts:

$\tau_1 = (-\infty, -NT)$ where N is a large integer;
$\tau_2 = (-NT, +NT)$; 2N cycles with period T;
$\tau_3 = (+NT, \infty)$

For $t \in \tau_1$ the particle is starting to move;
for $t \in \tau_2$ the motion is periodic;
For $t \in \tau_3$ the particle is stopping.

.

$$\vec{\chi} = \left\{ \int_{\infty}^{-NT} + \int_{-NT}^{+NT} + \int_{+NT}^{+\infty} \right\}$$

$$\square \; dt\, \vec{v}(t)\;\; e^{i\,\omega\,t}\; e^{-i\,\hat{n}\,\square\,\vec{r}(t)\,/c}$$

$$\vec{\chi} = \vec{\chi}_{start} + \vec{\chi}_{stop} \; +$$

$$\square \; \int_{-NT}^{+NT} dt\, \vec{v}(t)\;\; e^{i\,\omega\,t}\; e^{-i\,(\omega/c)\,\hat{n}\,\square\,\vec{r}(t)}$$

$$\vec{\chi} = \vec{\chi}_{start} + \vec{\chi}_{stop} \; + \vec{\chi}_N$$

$$\vec{\chi}_N = \sum_{n=-N}^{N-1} \int_{nT}^{(n+1)\,T} dt\, \vec{v}(t)$$

$$\square \; e^{i\,\omega\,t}\; e^{-i\,k\,\hat{n}\,\square\,\vec{r}(t)}$$

The precise way in which the motion starts and stops is not important, but χ_{start} and χ_{stop} must be finite and independent of N; 2N is the number of periodic cycles.

Evaluation of $\vec{\chi}$ for the periodic motion, τ_2.

Consider the cycle from t = nT to t = (n+1) T;

n = –N , –N+1, –N+2 , ... , N–2, N–1.

Now let *t'* = t – nT.

Then 0 ⩽ *t'* ⩽ T.

Change the integration over this cycle into the integral over *t'* from 0 to T. The point is, *all the integrals are equal* because of the periodicity,

$\vec{r}$(t'+nT) = $\vec{r}$(t') and $\vec{v}$(t'+nT) = $\vec{v}$(t').

Result

$$\vec{\chi}_N = \left(\sum_{n=-N}^{N-1} e^{i(\omega T)n} \right)$$

$$\square \int_0^T dt' \, \vec{v}(t') \, e^{i\omega\left(t' - k\,\hat{n} \bullet \vec{r}(t')\right)}$$

The integral does not depend on n.

So far we have

In[1356]:= scan82

Out[1356]=

$$\frac{d^2E}{d\omega d\Omega} = \frac{e^2\omega^2}{4\pi^2c^3}\left|\hat{n}\times\left(\vec{\chi}_{start}+\vec{\chi}_{stop}+\vec{\chi}_N\right)\right|^2 \qquad (221)$$

$$\frac{d^2P(\omega)}{d\omega\, d\Omega} = \lim_{N\to\infty} \frac{1}{2NT}\left(\frac{d^2E}{d\omega\, d\Omega}\right)$$

$\vec{\chi}_{start}$ and $\vec{\chi}_{stop}$ are independent of N so they drop out of the limit.

$$\frac{d^2P(\omega)}{d\omega\, d\Omega} = \frac{e^2\omega^2}{4\pi^2c^3} Q(\omega) \left|\hat{n}\times\int_0^T dt\, \vec{v}(t)\, e^{i\omega t}\, e^{-i(\omega/c)\hat{n}\cdot\vec{r}(t)}\right|^2$$

$$Q(\omega) = \lim_{N\to\infty}\frac{1}{2NT}\sum_{n,n'=-N}^{N-1} e^{i\omega T(n-n')} \qquad (224)$$

The calculation of Q(ω)

$Q(\omega) = \lim_{N\longrightarrow\infty}$

$$\frac{1}{2\,NT} \sum_{n,n'=-N}^{N-1} e^{i\,\omega T\,(n-n')}$$

In the limit $N \longrightarrow \infty$, $Q(\omega)$ is a '*periodic delta function*' .

The final result is,

$$Q(\omega) = \frac{\omega_0^2}{2\,\pi} \sum_{m=1}^{\infty} \delta(\omega - m\,\omega_0)$$

where $\omega_0 = \frac{2\,\pi}{T}$.

So, the spectrum of waves produced by the periodic motion of the particle is *discrete : it consists of all harmonics of the fundamental frequency* $\omega_0 = 2\pi/T$; i.e., $\omega = \omega_0, 2\,\omega_0, 3\,\omega_0, 4\,\omega_0$, ...

Powers radiated into the various harmonics

We have this result

$$\frac{d^2 P(\omega)}{d\omega\, d\Omega} = \Sigma_{m=1}^{\infty} \frac{e^2\,\omega_0^4\,m^2}{(2\,\pi\,c)^3}\,\delta(\omega - m\omega_0)$$

$$\square \left|\hat{n} \times \int_0^T dt'\,\vec{v}(t')\; e^{i\,m\omega_0\,t'}\, e^{-i\,m\,k_0\,\hat{n}\bullet\vec{r}(t')}\right|^2$$

Or,

$$\frac{d^2 P(\omega)}{d\omega\, d\Omega} = \Sigma_{m=1}^{\infty}\,\delta(\omega - m\omega_0)\,\frac{dP_m}{d\Omega}$$

where

$$\frac{dP_m}{d\Omega} = \frac{e^2\,\omega_0^4\,m^2}{(2\,\pi\,c)^3}$$

$$\square \left|\hat{n} \times \int_0^T dt'\,\vec{v}(t')\, e^{i\,m\omega_0\,t'}\, e^{-i\,m\,k_0\,\hat{n}\bullet\vec{r}(t)}\right|^2$$

= angular distribution of power radiated to the m-th harmonic.

Each m value behaves as an independent harmonic source

WT also derive these results:

$$\frac{dP_m}{d\Omega} = \frac{(m\omega_0)^2\, r^2}{8\,\pi\, c} \; | \; \hat{n} \times \vec{A}_m(\vec{x}) \; |^2$$

where

$$\vec{A}_m(\vec{x}) = \frac{e^{i\, m\, k_0\, r}}{c\, r} \int d^3 x' \, \vec{J}_m(\vec{x}') \, e^{-i\, m\, k_0\, \hat{n} \bullet \vec{x}'}$$

and

$$\vec{J}_m(\vec{x}) = \frac{e\, \omega_0}{\pi} \int_0^T dt \;\; \vec{v}(t) \; \delta(\vec{x} - \vec{r}(t)\;) \; e^{i\, m\omega_0\, t}$$