In[47]:=
```
dir0 = "/Users/OurMacBookAir/Documents";
dir1 =
  "/Teaching.2018.current/chapter11.current
    /RadLecs.999";
SetDirectory[StringJoin[dir0, dir1]];
FileNames[];
sc1 = Show[Import["fig11.9.png", "png"],
   ImageSize → 768];
sc2 = Show[Import["eq11.166.png", "png"],
   ImageSize → 768];
sc3 = Show[Import["fig11.10.png", "png"],
   ImageSize → 768];
sc4 = Show[Import["eq11.169.png", "png"],
   ImageSize → 768];
```

Wednesday October 24

Chapter 13: Relativistic Electrodynamics

Section 13.7: Relativistic kinematics in the context of linear and circular particle accelerators

1.

1⟹ The power radiated by an accelerating particle

Larmor's formula (see Section 11.6)

In the "instantaneous rest frame" of the particle, the radiated power is

$$P(t) = \frac{2\,e^2}{3\,c^3}\left[\vec{a}(t_0)\right]^2$$

(13.99)

$t_0 = t - r/c$

The Liénard result (see Section 11.6)

In an arbitrary reference frame,

$$P_L(t_r) = \frac{2\,e^2}{3\,c^3}\left[\frac{\dot{\vec{v}}^2 - \left(\vec{\beta}\times\dot{\vec{v}}\right)^2}{\left(1-\beta^2\right)^3}\right]_{\text{ret}}$$

(11.160)

$t_r = t - |\, x - r \,|^2$

WT point out that the derivation of

(11.160) — based on the Liénard-Wiechert potentials — “took several pages to derive.”

Using Special Relativity we can derive (11.160) trivially, based on the following theorem. *[See Exercise 13.7.1.]*

Theorem. $P(t_r)$ is a scalar with respect to Lorentz transformations.

Proof. It can be shown that

$$P_L(t_r) = \frac{2\,e^2}{3\,m^2\,c^3}\left(-\frac{dp^{\mu}}{d\tau}\,\frac{dp_{\mu}}{d\tau}\right)$$

(13.100)

which is manifestly a scalar.

Or, it makes sense because P = dE/dt and both E and t are time components of 4-vectors (x^{μ} and p^{μ}).

2.

2 $\Longrightarrow$ Radiation when $\vec{a} \parallel \vec{v}$ and when $\vec{a} \perp \vec{v}$

Electromagnetic waves carry energy and momentum. Therefore there is ***energy loss via radiation*** when a particle undergoes acceleration. We know formulas for the energy loss, from Section 11.6.

In[43]:= sc1

sc2

sc3

sc4

Out[43]=

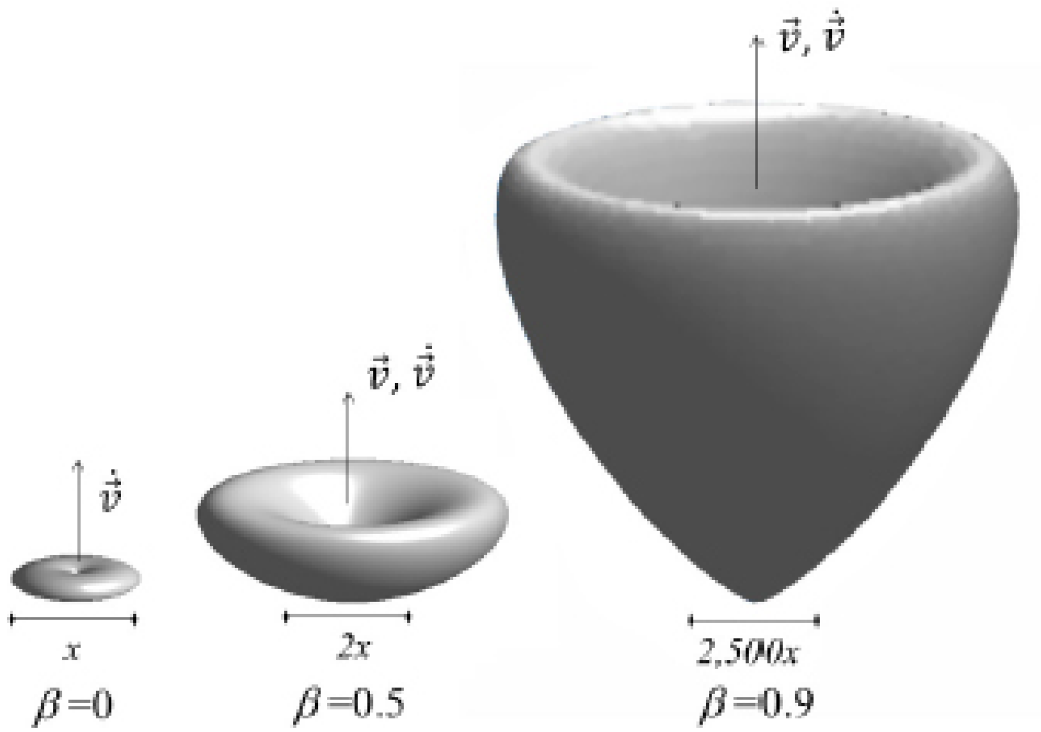

Fig. 11.9 Radiation patterns when $\dot{\vec{v}} \| \vec{v}$, for three different values
intervals below each pattern indicate the relative scales for
distributions shown.

Out[44]=

$$P(t_r) = \frac{2}{3}\frac{e^2}{c^3}\frac{\dot{v}^2}{(1-\beta^2)^3} = \frac{2}{3}\frac{e^2}{c^3}\gamma^6$$

Out[45]=

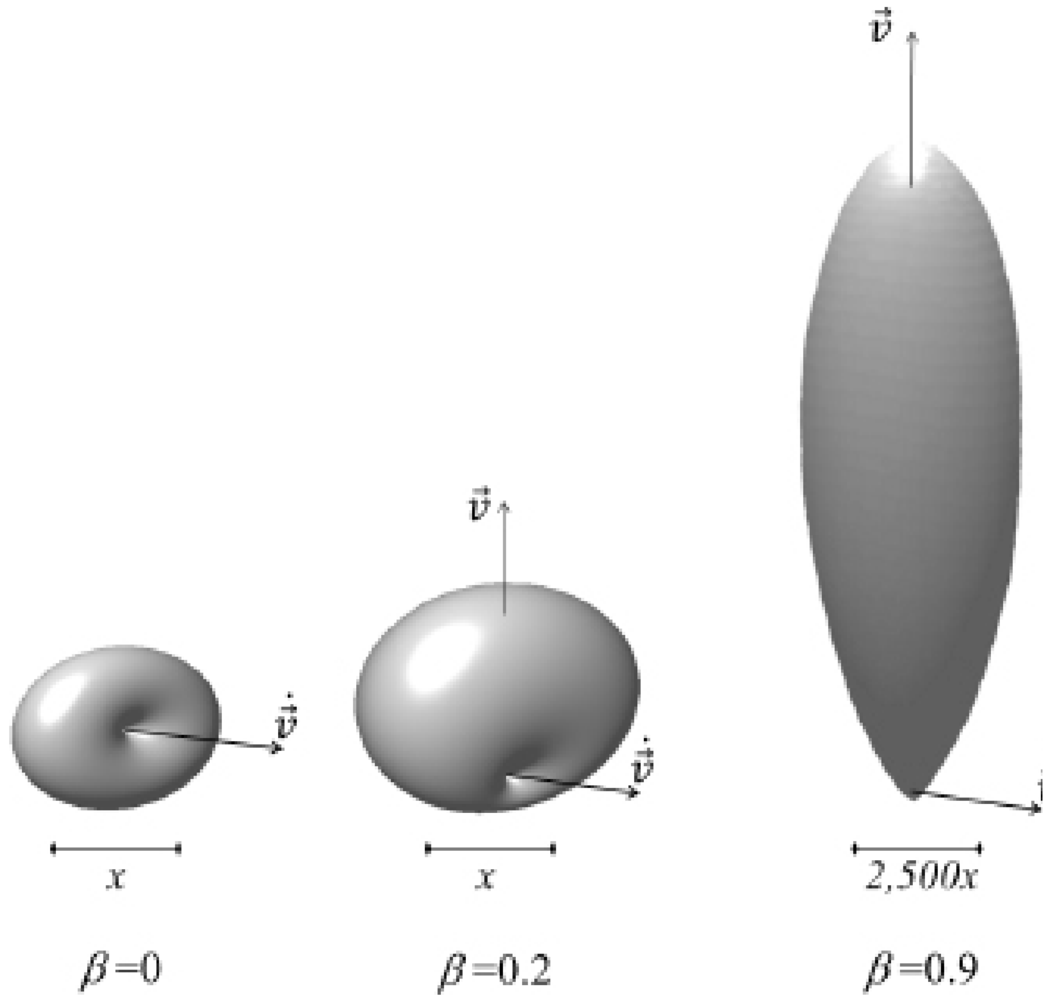

Fig. 11.10 Radiation patterns when $\dot{\vec{v}} \perp \vec{v}$, for three different va
The intervals below each pattern indicate the relative scales fo
distributions shown.

Out[46]=

$$P(t_r) = \frac{2e^2}{3c^3}\frac{\dot{v}^2}{(1-\beta^2)^3}(1-\beta^2) = \frac{2e^2}{3c^3}\frac{\dot{v}^2}{(1-\beta^2)^2} = \frac{2}{3}\frac{e^2}{c^3}\dot{v}^2\gamma^4. \qquad ($$

These results are the radiative losses for given parallel and perpendicular *accelerations*.

But what is more relevant is calculate the the radiative losses for parallel and perpendicular *forces*.

3.

Linear accelerators : $\vec{a} \parallel \vec{v}$

From Chapter 11,

$P_{\text{lin}} = \frac{2\,e^2}{3\,c}\,\gamma^6\left(\dot{\beta}\right)^2$ $\propto$ *γ^6 for given acceleration*

But now re-express P_{lin} in terms of $d\vec{p}/dt$. Relativistic dynamics of a particle (mass m) $E = \gamma\,mc^2$ and $\vec{p} = \gamma\,m\vec{v}$ where $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v/c$.

■■

$$\beta^2 = 1 - \gamma^{-2} = 1 - \left(\frac{\mathrm{mc}^2}{E}\right)^2$$

$$2\beta\,\dot{\beta} = 2\ m^2\ c^4\ E^{-3}\ \dot{E}$$

$$\left(\dot{\beta}\right)^2 = \left(\frac{m^2\ c^4\ \dot{E}}{\beta\, E^3}\right)^2 = \frac{\dot{E}^2}{\beta^2}\ \frac{(\mathrm{mc}^2)^4}{(\gamma \mathrm{mc}^2)^6}$$

$$\therefore\ \gamma^6 \left(\dot{\beta}\right)^2 = \frac{\dot{E}^2}{\beta^2\,(\mathrm{mc}^2)^2}$$

■■

$$E^2 = p^2\, c^2 + m^2\, c^4$$

$$2\mathrm{E}\, \dot{E} = 2\mathrm{p}\, \dot{p}\, c^2$$

$$\dot{E} = \dot{p}\, \frac{\mathrm{pc}^2}{E} = \dot{p}\, \frac{\gamma \mathrm{mv}\, c^2}{\gamma \mathrm{mc}^2} = \dot{p}\, \mathrm{v}$$

$$\therefore \gamma^6 \left(\dot{\beta}\right)^2 = \frac{\dot{p}^2\, v^2}{(v/c)^2\, (\mathrm{mc}^2)^2} = \left(\frac{\dot{p}}{\mathrm{mc}}\right)^2$$

$$P_{\mathrm{lin}} = \frac{2\, \mathrm{e}^2}{3\, c}\, \gamma^6 \left(\dot{\beta}\right)^2 = \frac{2\, \mathrm{e}^2}{3\, m^2\, c^3} \left(\frac{d\, \vec{p}}{\mathrm{dt}}\right)^2$$

(13.102)

In terms of $d\,\vec{p}/\mathrm{dt}$ ($\equiv$ the force), $P_{\mathrm{lin}} \propto \gamma^0$.

4.

Circular accelerators : $\vec{a} \perp \vec{v}$

From Chapter 11,

$P_{\text{circ}} = \frac{2\,e^2}{3\,c}\,\gamma^4\left(\dot{\beta}\right)^2$ $\propto \gamma^4$ *for given acceleration*

But now re-express P_{circ} in terms of $d\vec{p}/\text{dt}$.

■■

For circular motion,

$|\,\vec{v}\,|$ is constant, $\text{v} = \text{R}\omega_c$;

also, $d\,\vec{v}\,/\,\text{dt}$ is perpendicular to $\vec{v}$.

$$\vec{p} = \gamma\, m\, \vec{v} \quad \Longrightarrow \quad \frac{d\,\vec{p}}{\text{dt}} = \gamma\,\text{m}\,\frac{d\,\vec{v}}{\text{dt}}$$

$$\left(\dot{\beta}\right)^2 = \left(\frac{1}{c}\,\frac{d\,\vec{v}}{\text{dt}}\right)\cdot\left(\frac{1}{c}\,\frac{d\,\vec{v}}{\text{dt}}\right) = \frac{\left(\dot{p}\right)^2}{(\gamma\,\text{mc})^2}$$

$$P_{\text{circ}} = \frac{2\,e^2}{3\,c}\,\gamma^4\left(\dot{\beta}\right)^2 = \frac{2\,e^2}{3\,m^2\,c^3}\,\gamma^2\left(\frac{d\,\vec{p}}{\text{dt}}\right)^2$$

(13.104)

In terms of $d\,\vec{p}/\text{dt}$ (≡ the force), $P_{\text{circ}} \propto \gamma^2$.

"Under the same force, the radiative losses from circular acceleration are a factor of γ^2 times larger than those from linear acceleration."

5.

3 $\Longrightarrow$ Example of a linear accelerator; e.g., SLAC

A charged particle moves in a constant electric field. Then $|d\vec{p}/\mathrm{dt}|$ is constant. [See Exercise 13.7.3.] Let T be the time of flight. Then the total radiated energy is

$$E_{\mathrm{rad}} = \frac{2\,e^2}{3\,m^2\,c^3}\left(\frac{\mathrm{dp}}{\mathrm{dt}}\right)^2 \mathrm{T}$$

$$E^2 = p^2\,c^2 + m^2\,c^4$$

Interesting exercise: $\frac{\mathrm{dE}}{\mathrm{dx}} = \frac{\mathrm{dp}}{\mathrm{dt}}$

$$E_{\mathrm{rad}} = \frac{2\,e^2}{3\,m^2\,c^3}\left(\frac{\mathrm{dE}}{\mathrm{dx}}\right)^2 T$$

We may say $T \approx L/c$ for a linear accelerator with length L,

so the energy supplied is

$$E_{\text{supplied}} = \frac{dE}{dx} L \approx \frac{dE}{dx} T c$$

Thus,

$$\frac{E_{\text{rad}}}{E_{\text{supplied}}} = \frac{2 e^2}{3 m^2 c^4} \left(\frac{dE}{dx} \right)$$

WT give these parameters:

dE/dx = 10 MeV /m

electron: $E_{\text{rad}} / E_{\text{supplied}} \sim 10^{-14}$

proton: $E_{\text{rad}} / E_{\text{supplied}} \sim 10^{-20}$

In a linear accelerator, radiation is not a significant source of energy loss.

6.

4 ⟹ Example of a circulator accelerator; e.g., LHC

From Section 11.9, Eq. (11.171), the energy supplied for a single cycle is

$$(\Delta E)_{cycle} = \frac{4\pi e^2}{3R} \frac{\beta^3}{(1-\beta^2)^2}$$

$$= \frac{4\pi e^2}{3R} \left(\frac{E}{mc^2}\right)^4 \beta^3$$

Take the LHC as an example. Parameters:

$R = 4.3$ km $= 4.3 \times 10^5$ cm;

proton energy = E = 7 TeV (= design energy)

∴

$(\Delta E)_{cycle} = 4.5$ keV $= 4.5 \times 10^3$ eV

energy radiated per proton per revolution

For comparison, what?
The number of revolutions that would produce radiation of 7 TeV:
$\#r = 7\ \text{TeV} / (\Delta E)_{\text{cycle}} = 1.5 \times 10^9$
(assuming the particle energy is maintained constant)
period of revolution = $2\pi R/c = 10^{-4}$ sec;
time for #r revolutions = 1.5×10^5 sec = 1.7 days.
Evidently, radiation losses are not a limiting factor at the LHC.

<u>Magnetic field requirement</u>

●For a nonrelativistic proton,
$mR\omega^2 = (e/c)\, R\omega\, B$ and $p = mv = mR\omega$;

$$\Longrightarrow R = \frac{p}{m\omega} = \frac{p}{m(eB/mc)} = \frac{pc}{eB}$$

●For an ultrarelativistic proton, the same relation is true [Exercise 13.7.4];

also, E = pc; so

$$\Longrightarrow \; B = \frac{E}{eR} \,.$$

●For the LHC, E = 7 TeV

$\Longrightarrow$ B = 5.4 × 10^4 Gauss = 5.4 Tesla.

(Actually B > 5.4 Tesla because the tunnel is not totally filled with bending magnets.)

An electron synchrotron is different!

Exercise 13.7.5

7.

Homework Assignment #11 (due Nov 2) will include

Exercise 13.7.1

Exercise 13.7.2

Exercise 13.7.3

Exercise 13.7.4

Exercise 13.7.5