Friday Oct 26
The classical electron model of light scattering

● An electromagnetic wave impinges upon a molecule

The electrons experience electric and magnetic forces.

The electric force is much stronger than the magnetic force, because atomic electrons are nonrelativistic ($v \ll c$).

We'll assume that the incoming E.M. wave is a plane wave, moving in the z direction, and linearly polarized in the x direction;

$$\vec{E}(\vec{x},t) = \hat{e}_x E_0 \, e^{i(kz-\omega t)}$$

Re implied!

● Motion of the electron(s)

Using the classical electron model, the motion of an electron is described by

$$m \frac{d^2\vec{r}}{dt^2} = -K\vec{r} - \gamma \frac{d\vec{r}}{dt} - e E_0 \hat{e}_x e^{-i\omega t}$$

We can treat the electric field as uniform in space because the wavelength is $\gg$ the size of the atom.

(Really this *model equation* represents the motion of the negative charge in the atom, which may involve many electrons. But I'll continue to call it "the electron".)

The equation: a damped driven oscillator.

The steady state solution: the electron oscillates in the x direction, with the driving frequency ω;

$$\vec{r}(t) = \hat{e}_x \, e^{-i\omega t} \, A$$

where

$$A = \frac{-e E_0}{m(\omega_0^2 - \omega^2) - i\gamma\omega}$$

$\omega_0 = \sqrt{K/m}$ = the “natural frequency”.

● The electron radiates

Larmor’s formula; the average power integrated over all directions is

$$P_{avg} = \frac{2 e^2 a^2}{3 c^3} \text{ where } a^2 = \left\langle (\ddot{x})^2 \right\rangle.$$

We need to be careful about taking the real part of x(t).

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We have $A = A_1 + i A_2$, and so

$x(t) = \text{Re}\{ e^{-i\omega t} A \} = A_1 \cos(\omega t) + A_2 \sin(\omega t)$

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$$\left\langle \ddot{x}^2 \right\rangle .= \frac{1}{2} \omega^4 (A_1^2 + A_2^2)$$

$$P_{avg} = \frac{e^2}{3 c^3} \omega^4 (A_1^2 + A_2^2)$$

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(* Calculations *)
A = -e * E0 / (m (ω0 ^ 2 - ω ^ 2) - I * γ * ω)
{A1, A2} =
  {Re[A], Im[A]} // ComplexExpand // Simplify
A1 ^ 2 + A2 ^ 2 // Expand // Simplify
Pavg = e ^ 2 * ω ^ 4 / (3 * c ^ 3) * (%)
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Out[788]= $-\frac{e\ E0}{-\mathrm{i}\ \gamma\ \omega + m\ \left(-\omega^2 + \omega0^2\right)}$

Out[789]= $\left\{\frac{e\ E0\ m\ \left(\omega^2 - \omega0^2\right)}{\gamma^2\ \omega^2 + m^2\ \left(\omega^2 - \omega0^2\right)^2},\ -\frac{e\ E0\ \gamma\ \omega}{\gamma^2\ \omega^2 + m^2\ \left(\omega^2 - \omega0^2\right)^2}\right\}$

Out[790]= $\frac{e^2\ E0^2}{\gamma^2\ \omega^2 + m^2\ \left(\omega^2 - \omega0^2\right)^2}$

Out[791]= $\frac{e^4\ E0^2\ \omega^4}{3\ c^3\ \left(\gamma^2\ \omega^2 + m^2\ \left(\omega^2 - \omega0^2\right)^2\right)}$

● Scattering and the cross section

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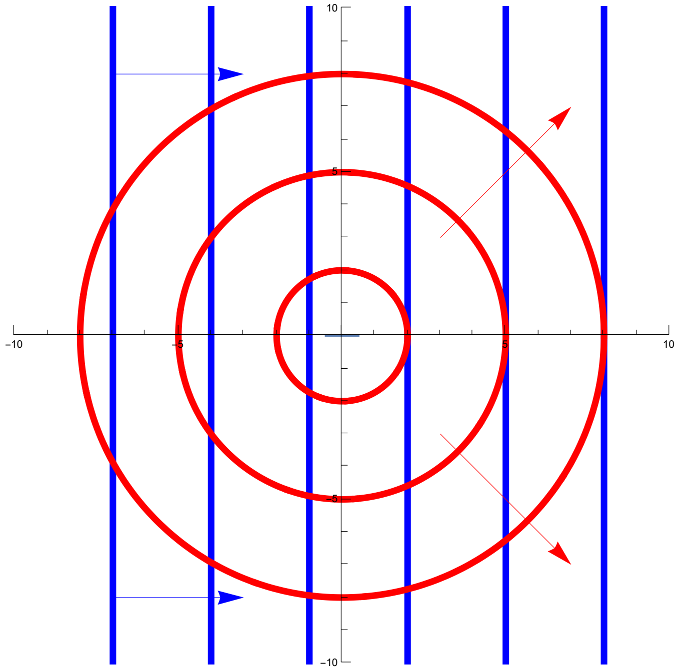
Out[792]=



The definition of the scattering cross section is

$$\sigma = \frac{P}{S_{inc}}$$

where P = outgoing power averaged over time;

and S_{inc} = incoming intensity ≡ power per unit

area averaged over time

Of course we know S_{inc}

$$S_{inc} = \frac{c}{4\pi} \langle \vec{E} \times \vec{B} \rangle = \frac{c}{8\pi} E_0^2$$

Thus,

$$\sigma = \frac{e^4 E_0^2 \omega^4}{3 c^3} \frac{1}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \frac{8\pi}{c E_0^2}$$

$$\sigma = \frac{8\pi}{3} r_e^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega/m)^2}$$

$$r_e = \frac{e^2}{mc^2}$$ = the "classical radius" of the electron

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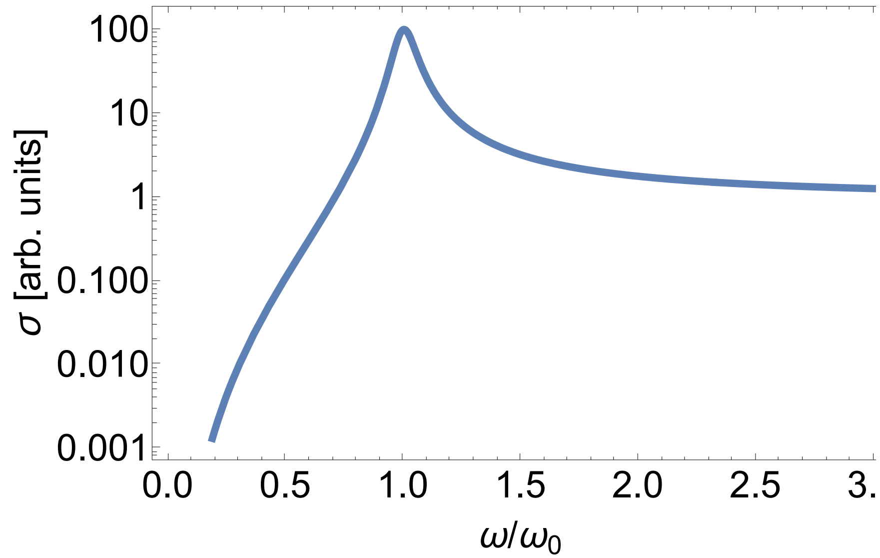

● Three regions of frequency

■■ For $\omega \ll \omega_0$
■ $\sigma \propto \omega^4$
■ Rayleigh scattering
■ Why the sky is blue and sunsets are red

■■ For $\omega \sim \omega_0$
■ resonant scattering
■ The frequency of ligh is near the natural frequency $\Longrightarrow$ a large peak in the cross section.
■ Resonance fluorescence in atomic physics (quantum theory)

■■ For $\omega \gg \omega_0$
■ $\sigma = 8\pi\, r_e^2 /3$, a constant = 66.5 fm^2 = 665 millibarn
■ Thomson scattering
■ This is the scattering of light from a free electron ($\omega_0 = \gamma = 0$) for $\hbar\omega \gg 1$ eV.
■ Applications in plasma physics

● A fourth region of frequency

■■ For $\hbar\omega \gtrsim mc^2 = 0.5$ MeV = hard X-rays
■ $\Longrightarrow$ Compton scattering
■ The frequency of the scattered waves is not ω.
■ $\omega' < \omega$ in the target electron rest frame
■ ω' depends on θ
■ Compton's experiments (early 1920's) — the definitive proof of the photon theory of light.
■ $\gamma + e \longrightarrow \gamma' + e'$
■ QED is necessary to calculate the cross section; the Klein-Nishina formula.

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Out[794]=