Mon Oct 29

Chapter 12: Scattering and Diffraction

Last time: Light scattering from an electron

- Rayleigh scattering (γ + bound e)
- Resonant scattering
- Thomson scattering (γ + free e)

Today: Section 12.1: The polarized scattering cross section

Consider harmonic waves incident on a charge density;

$\vec{E}_{in}(\vec{x},t) = \vec{E}(\vec{x})\, e^{-i\omega t}$

and $\vec{B}_{in}(\vec{x},t) = \vec{B}(\vec{x})\, e^{-i\omega t}$

<u>Field equations</u>

$\vec{E}$ and $\vec{B}$ mean $\vec{E}(\vec{x})$ and $\vec{B}(\vec{x})$;

$\vec{J}$ means $\vec{J}(\vec{x})$.

$\nabla \times \vec{B} = -ik\,\vec{E} + (4\pi/c)\,\vec{J}$ where $k = \omega/c$

$\nabla \times \vec{E} = ik\,\vec{B}$

Combine the field equations ,

$$\nabla \times (\nabla \times \vec{E}) = k^2 \vec{E} + ik \frac{4\pi}{c} \vec{J}$$

and

$$ik\ \nabla \cdot \vec{E} = \frac{4\pi}{c} \nabla \cdot \vec{J}$$

So,

$$-(\nabla^2 + k^2)\vec{E} =$$

$$= ik \frac{4\pi}{c} \{ \vec{J} + \frac{1}{k^2} \nabla(\nabla \cdot \vec{J}) \}$$

↖ *Wave equation with a source;*
a linear inhomogeneous eq.

Incident *plane* waves have

$\vec{E}_{\mathrm{inc.}}(\vec{x}) = E_0\, \hat{e}_0\, e^{i \vec{k}_0 \cdot \vec{x}}$ where $\hat{e}_0 \cdot \vec{k}_0 = 0.$

$\vec{k}_0$ = wave vector

$\hat{e}_0$ = polarization vector (∈ transverse plane)

The Green's function for this problem

To solve the wave equation, recall the retarded Green's function $G(\vec{x},t;\vec{x}',t')$:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G$$

$$= -4\pi\,\delta^3(\vec{x}-\vec{x}')\,\delta(t-t');$$

$$G = \frac{1}{R}\,\delta(t - t' - R/c) \quad \text{where } R = |\vec{x}-\vec{x}'|$$

Fourier transform w.r.t. time,

$$(\nabla^2 - k^2)\,G^{\text{outgoing}} = -4\pi\,\delta^3(\vec{x}-\vec{x}');$$

$k = \omega/c$

where $G^{\text{out}}(\vec{x},\vec{x}';k)$ = F.T. of G w.r.t. time.

$$G^{\text{out}}(\vec{x},\vec{x}';k) = \text{F.T. of } G$$

$$= \int_{-\infty}^{\infty} d(t-t')\, e^{i\omega(t-t')}\,\frac{1}{R}\,\delta(t-t'-R/c)$$

$$= \frac{e^{ikR}}{R}$$

Note: G^{out} is an outgoing wave from a point source

$$G^{out}(\vec{x}, \vec{x}\,'; k) = \frac{e^{ikR}}{R} \quad \text{where } R = |\vec{x} - \vec{x}\,'|$$

⇒ The solution of the wave equation

$$\vec{E} = \vec{E}_{inc.} +$$

$$\frac{ik}{c} \int d^3x'\, G^{out}(\vec{x}, \vec{x}\,'; k)$$

$$\cdot \left\{ \vec{J} + \frac{1}{k^2} \nabla' \left(\nabla' \cdot \vec{J} \right) \right\}$$

integrate by parts

$$= \vec{E}_{inc.} + ik \left(1 + \frac{1}{k^2} \nabla\nabla\right) \cdot \vec{A}(\vec{x})$$

dyadic

$$\vec{A}(\vec{x}) = \frac{1}{c} \int d^3x' \frac{e^{ik|\vec{x} - \vec{x}\,'|}}{|\vec{x} - \vec{x}\,'|} \vec{J}(\vec{x}\,')$$

The far zone approximation

$$|\vec{x} - \vec{x}\,'| \approx r - \hat{n} \cdot \vec{x}\,'$$

where $\hat{n}$ is the direction of $\vec{x}$; this gives the outgoing scattered wave

$$\vec{A}_{sc}(\vec{x}) = \frac{e^{ikr}}{cr} \int d^3x'\, \vec{J}(\vec{x}\,')\, e^{-ik\hat{n} \cdot \vec{x}\,'}$$

Also, $\nabla \longrightarrow ik\hat{n}$

So the $\vec{E}$ field of the scattered wave is

$$\vec{E}_{sc} = ik\,(1 - \hat{n}\hat{n}) \cdot \vec{A}_{sc}$$

$$= -ik\,\hat{n} \times (\hat{n} \times \vec{A}_{sc})$$

The scattered power

The asymptotic fields,

$$\vec{E} = \vec{E}_{inc} + \vec{E}_{sc}$$

$$\& \quad \vec{E}_{sc} = -ik\,\hat{n} \times (\hat{n} \times \vec{A}_{sc})$$

$$\vec{B} = \vec{B}_{inc} + \vec{B}_{sc}$$

$$\& \quad \vec{B}_{sc} = -ik\,\hat{n} \times \vec{A}_{sc}$$

So the *time-averaged scattered power* is

$$\frac{dP}{d\Omega} = \frac{r^2 c}{8\pi} |\vec{E}_{sc}|^2 = \frac{r^2 c}{8\pi} |\vec{B}_{sc}|^2$$

$$= \frac{k^2 r^2 c}{8\pi} |\hat{n} \times \vec{A}_{sc}|^2$$

Polarizations of the scattered waves

We can separate the outgoing waves into linear or circular polarizations.

(We did not do this back in Chapter 11 for radiation fields, but we could have. In fact the same method applies to either scattering or radiation. For example, synchrotron radiation is generally polarized—which can be used in radio astronomy.)

We have $\vec{E}_{sc} \cdot \hat{n} = 0$. For linear polarization, define two orthogonal directions both perpendicular to $\hat{n}$; denote the unit vectors by $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$, with $\hat{\epsilon}_1 \times \hat{\epsilon}_2 = \hat{n}$. Then

$$\vec{E}_{sc} = \sum_{f=1}^{2} \hat{\epsilon}_f \, (\hat{\epsilon}_f \cdot \vec{E}_{sc}).$$

The power distribution is $\frac{dP}{d\Omega} = \sum_f \left(\frac{dP}{d\Omega}\right)_f$

where

$$\left(\frac{dP}{d\Omega}\right)_f = \frac{r^2 c}{8\pi} \, | \, \hat{\epsilon}_f \cdot \vec{E}_{sc} \, |^2 .$$

Define the scattering *amplitude* $\vec{f}\left(\vec{k}, \vec{k}_0\right)$

Write $\vec{E}_{sc} = \frac{e^{ikr}}{r} E_0 \, \vec{f}\left(\vec{k}, \vec{k}_0\right)$.

$\vec{k}_0$ = incident wave vector ;

$\vec{k} = \mathrm{k} \, \hat{n}$; $\mathrm{k} = \omega / c$.

Then

$$\left(\frac{dP}{d\Omega}\right)_f = \frac{c \, |E_0|^2}{8\pi} \, | \, \hat{\epsilon}_f \cdot \vec{f}(\vec{k}, \vec{k}_0) \, |^2$$

and

$$\hat{\epsilon}_f \cdot \vec{f}(\vec{k}, \vec{k}_0) =$$

$$= \frac{ik}{cE_0} \, \hat{\epsilon}_f \cdot \int d^3x' \, \vec{J}(\vec{x}') \exp\{-i\vec{k} \cdot \vec{x}'\}$$

The polarized scattering cross section

Suppose the incident wave has a definite polarization $\hat{e}_i$.
The scattered wave will have both outgoing polarizations, but let's assume that what is observed is the polarization $\hat{e}_f$. The cross section for this process is $(d\sigma/d\Omega)_{f,i}$.

(element of a 2 x 2 matrix)

$$\left(\frac{d\sigma}{d\Omega}\right)_{f,i} = \left| \hat{e}_f \cdot \vec{f}_i(\vec{k}, \vec{k}_0) \right|^2$$

$$= \frac{k^2}{c^2 |E_0^2|}$$

$$\square \left| \hat{\epsilon}_f \cdot \int d^3x' \, \vec{J}_{(i)}(\vec{x}') \exp\{-i\vec{k} \cdot \vec{x}'\} \right|^2$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \text{average and sum}$$

$$= \frac{1}{2} \sum_{i,f=1}^{2} \left(\frac{d\sigma}{d\Omega}\right)_{f,i}$$

Next we need to calculate the current density $\vec{J}(\vec{x})$.

Remember, *the current density is created by the incident wave.*

Also, what kind of object is scattering the light?