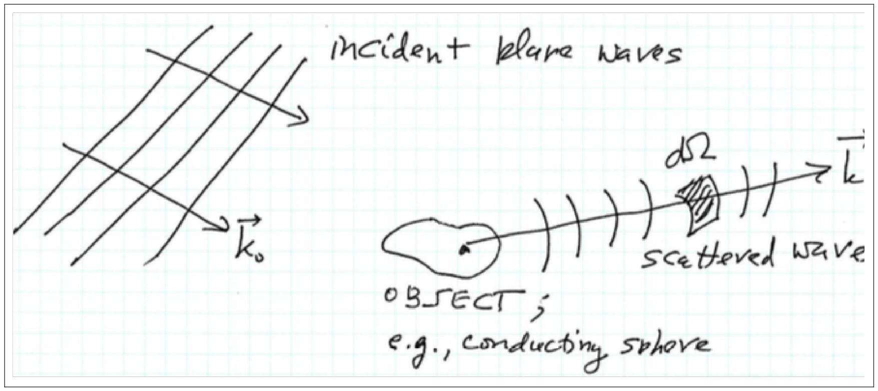
Chapter 12 - Scattering and Diffraction

Scattering

Plane waves incident on an object...

In[2275]:= scan1

Out[2275]=

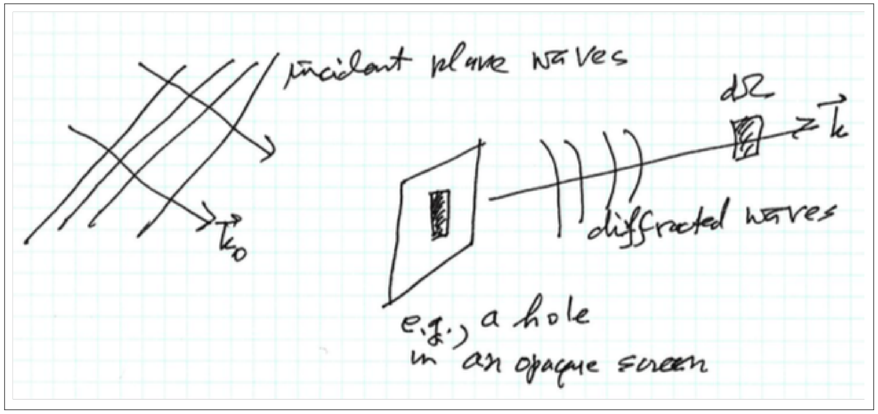


Diffraction

Plane waves incident on a hole in a screen...

In[2276]:= scan2

Out[2276]=



For both problems, the goal is to find the scattered fields in the far zone.

Wilcox and Thron: first scattering, then diffraction;

Jackson: first diffraction, then scattering

Review Section 12.1

In[2278]:= scan3

Out[2278]=

$$\vec{E} = \vec{E}_{inc} + \vec{E}_{sc}$$

$$\vec{E}_{inc}(\vec{x},t) = \hat{\epsilon}_i E_0 e^{i(\vec{k}_0\cdot\vec{x}-\omega t)}$$

$$\vec{E}_{sc}(\vec{x},t) = \vec{E}_{sc}(\vec{x})\, e^{-i\omega t}$$

$$\vec{E}_{sc}(\vec{x}) = \frac{e^{ikr}}{r} E_0 \vec{f}(\vec{k},\vec{k}_0) \quad \text{(far zone)}$$

Scattering amplitude

$$\vec{f}(\vec{k},\vec{k}_0) = \frac{ik}{cE_0}\int d^3x'\, \vec{J}(\vec{x}')\, e^{-i\vec{k}\cdot\vec{x}'}$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{f,i} = \left|\hat{\epsilon}_f \cdot \vec{f}_{(i)}(\vec{k},\vec{k}_0)\right|^2$$

Section 12.2
The Kirchhoff identity for scattering

The equation for the scattering amplitude $\vec{f}\left(\vec{k},\vec{k}_0\right)$ in Section 12.1 depends on the current density $\vec{J}(\vec{x}\,')$.

Gustav Kirchhoff had a different idea — express $\vec{f}(\vec{k},\vec{k}_0)$ in terms of the fields on the boundary surfaces.

Preliminaries

Green's first identity

In[2279]:= scan4

Out[2279]=

$$\int_V d^3x \left(\psi \nabla^2\phi + \nabla\psi \cdot \nabla\phi \right) = \oint_S da\, \psi\, \hat{n}\cdot\nabla\phi$$

Proof $\oint_S da\, \hat{n}\cdot(\psi\nabla\phi)$

$$= \int_V d^3x\, \nabla\cdot(\psi\nabla\phi) \quad \text{(Gauss theorem)}$$

$$= \int_V d^3x \left\{ \psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi \right\} \quad \checkmark$$

Green's second identity

In[2281]:= scan5

Out[2281]=

$$\int_V d^3x \left[\psi\nabla^2\phi - \phi\nabla^2\psi \right] = \oint_S da \left(\psi\,\hat{n}\cdot\nabla\phi - \phi\,\hat{n}\cdot\nabla\psi \right)$$

Proof Using (1)

$$\int_V d^3x \left[\psi\nabla^2\phi - \phi\nabla^2\psi + \nabla\psi\cdot\nabla\phi - \nabla\phi\cdot\nabla\psi \right]$$

$$= \oint_S da \left[\psi\,\hat{n}\cdot\nabla\phi - \phi\,\hat{n}\cdot\nabla\psi \right] \quad \checkmark$$

Notations

In[2282]:= `WT12p1(* FIGURE 12.1 *)`

Out[2282]=

First we use Green's second identity to obtain a new expression for the electric field. The Helmholtz equations for $\vec{E}_{\rm sc}$ and $G^{\rm out}$ are:

$$-(\vec{\nabla}'^2 + k^2)\vec{E}_{\rm sc}(\vec{x}\,') = \frac{4\pi}{c} ik \left(\vec{J}(\vec{x}\,') + \frac{1}{k^2}\vec{\nabla}'(\vec{\nabla}' \cdot \vec{J}(\vec{x}\,'))\right), \qquad (12.38)$$

$$(\vec{\nabla}'^2 + k^2) G^{\rm out}(\vec{x}, \vec{x}\,', k) = -4\pi\delta(\vec{x} - \vec{x}\,'). \qquad (12.39)$$

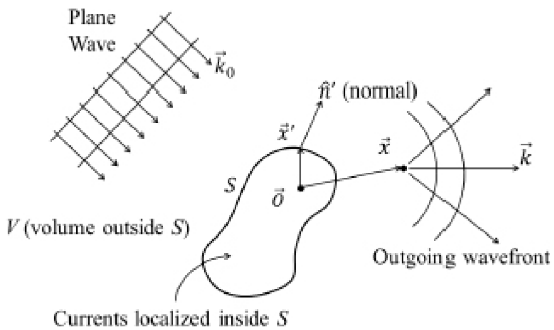


Fig. 12.1 Notation for derivation of Kirchhoff's identity.

After a long derivation, the cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{f,i} = \left| \hat{\epsilon}_f \cdot \vec{f}_{(i)}(\vec{k}, \vec{k}_0) \right|^2$$

$$= |Q|^2$$

and

$$Q = \frac{ik}{4\pi E_0} \oint_S da' \, e^{-i\vec{k}\cdot\vec{x}\,'}$$

$$\cdot \left\{ \hat{\epsilon}_f \cdot [\hat{n}' \times \vec{B}_{sc} + \hat{k} \times (\hat{n}' \times \vec{E}_{sc})] \right.$$

This determines the cross section, in terms of the scattered fields on the surface S. Recall, S is any surface that encloses the current.

Section 12.3 : Short wavelength scattering from a conducting sphere

In[2251]:= WT12p2

Out[2251]=

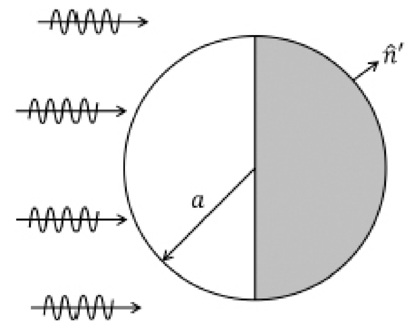


Fig. 12.2 Small-wavelength scattering from a conducting sphere, with shadow and illuminated side.