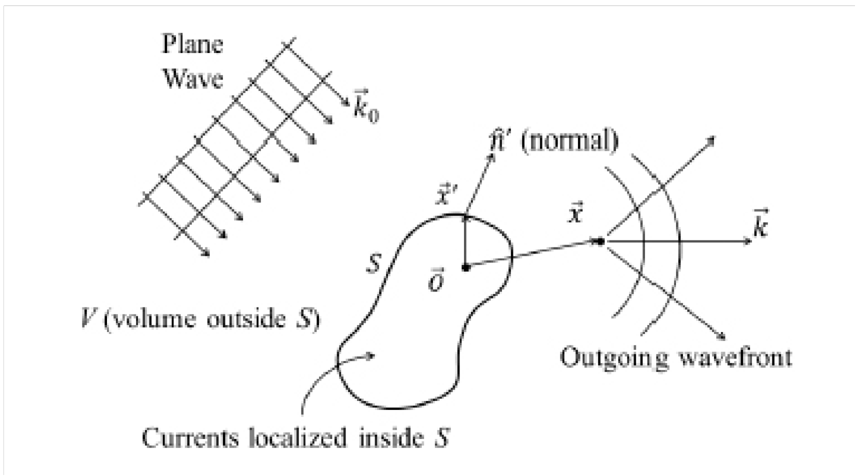
Section 12.3
Short wavelength scattering from a conducting sphere

See also, Jackson Section 10.10

In[2367]:= sc31

Out[2367]=



The Kirchhoff identity for scattering

$$\hat{\epsilon}_f \cdot \vec{f}\left(\vec{k}, \vec{k_0}\right) = \frac{ik}{4\pi E0} \oint_S da' \, e^{-i\vec{k}.\vec{x}'}$$

$$\cdot \left\{ \hat{n}' \times \vec{B}sc + \hat{k} \times \left(\hat{n}' \times \vec{E}_{sc} \right) \right\}$$

(12.49)

Short wavelength scattering

In[2368]:= sc32

Out[2368]=

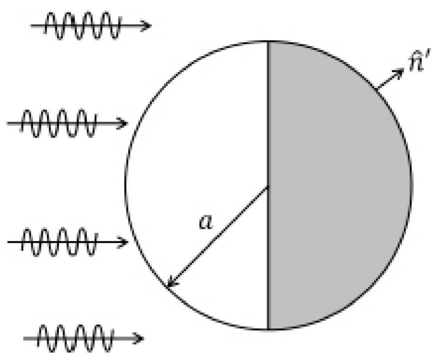


Fig. 12.2 Small-wavelength scattering from a conducting sphere, with shadow and illuminated side.

We write

$\vec{E}(\vec{x}) = \vec{E}_{inc}(\vec{x}) + \vec{E}_{sc}(\vec{x})$

$\vec{B}(\vec{x}) = \vec{B}_{inc}(\vec{x}) + \vec{B}_{sc}(\vec{x})$

Now to integrate over S, we'll impose some boundary conditions.

Exact boundary conditions

For all points on S,

- $\hat{n}' \cdot (\vec{B}_{inc} + \vec{B}_{sc}) = 0$
- $\hat{n}' \times (\vec{E}_{inc} + \vec{E}_{sc}) = 0$

Approximate boundary conditions

Because we don't know σ and $\vec{K}$ on S, we will make some approximations, valid for ka ≫ 1, based on physical observations.

$\lambda \ll$ a implies ka ≫1

- On S, on the shadow side, $\vec{E} = \vec{B} = 0$.

(shadow)

- On S, on the illuminated side, use the boundary conditions that correspond to *reflection from a flat conducting surface* (the law of equal angles, with Fresnel's ideas about polarization)

$$\hat{n}' \cdot \vec{E}_{sc} = \hat{n}' \cdot \vec{E}_{inc} \quad \textit{(illuminated)}$$

$$\hat{n}' \times \vec{B}_{sc} = \hat{n}' \times \vec{B}_{inc}$$

One might think a little more about this...

★ Thus we assume, as reasonable approximations for the limit $ka \gg 1$,

(54) $\vec{E}_{sc}(sh) \approx -\vec{E}_{inc}$

(55) $\vec{B}_{sc}(sh) \approx -\vec{B}_{inc}$

(56) $\vec{E}_{sc}(ill)$

$$\approx \hat{n}' (\hat{n}' \cdot \vec{E}_{inc}) + \hat{n}' \times (\hat{n}' \times \vec{E}_{inc})$$

(57) $\vec{B}_{sc}(ill)$

$$\approx -\hat{n}' (\hat{n}' \vec{B}_{inc}) - \hat{n}' \times (\hat{n}' \times \vec{B}_{inc})$$

The scattering amplitude

$$\hat{\epsilon}_f \cdot \vec{f} = \hat{\epsilon}_f \cdot \vec{f}_{sh} + \hat{\epsilon}_f \cdot \vec{f}_{ill}$$

$$\vec{f}_{sh} = \int_{S,\text{shadow side}}$$

$$\vec{f}_{ill} = \int_{S,\text{illuminated side}}$$

● Contribution from the *shadow side* of S

$$\hat{\epsilon}_f \cdot \overrightarrow{f_{sh}} = \frac{ik}{4\pi E_0} \int_{Ssh} da'\, e^{-i\vec{k}.\vec{x}'}\, \hat{\epsilon}_f \cdot$$

$$\cdot \{ \hat{n}' \times \overrightarrow{B_{sc}} + \hat{k} \times (\hat{n}' \times \overrightarrow{E_{sc}}) \}$$

$$= \frac{-ik}{4\pi E_0} \int_{Ssh} da'\, e^{-i\vec{k}.\vec{x}'}\, \hat{\epsilon}_f \cdot$$

$$\cdot \{ \hat{n}' \times \overrightarrow{B_{inc}} + \hat{k} \times (\hat{n}' \times \overrightarrow{E_{inc}}) \}$$

$$= \frac{-ik}{4\pi} \int_{Ssh} da'\, e^{-i\vec{k}.\vec{x}'}\, e^{+i\vec{k_0}.\vec{x}'}\, \hat{\epsilon}_f \cdot$$

$$\cdot \{ \hat{n}' \times \left(\hat{k}_0 \times \hat{\epsilon}_i\right) + \hat{k} \times (\hat{n}' \times \hat{\epsilon}_i) \}$$

- Note the factor exp{ i $(\vec{k}_0 - \vec{k}) \cdot \vec{x}'$ }.
- Note the incident polarization.
- $\hat{n}' \times \left(\hat{k}_0 \times \hat{\epsilon}_i\right)$ = k0(n'.ϵ)−ϵ(n'.k0)

$$= \hat{k}_0 (\hat{n}' \cdot \hat{\epsilon}_i) + \hat{k}_0 \times (\hat{n}' \times \hat{\epsilon}_i)$$

Result so far is $\hat{\epsilon}_f \cdot \overrightarrow{f_{sh}}$ =

$$= \frac{-ik}{4\pi} \int_{Ssh} da'\, e^{-i\left(\vec{k}-\vec{k_0}\right).\vec{x}'}\, \hat{\epsilon}_f \cdot$$

$$\cdot \{ \hat{k}_0 (\hat{n}' \cdot \hat{\epsilon}_i) + (\hat{k}_0 + \hat{k}) \times (\hat{n}' \times \hat{\epsilon}_i) \} \qquad (12.60)$$

Because ka ≫1 we can approximate the integral, as follows. The factor $\exp\{-i(k-k_0).x'\}$ oscillates rapidly as a function of $\vec{x}'$, unless $\vec{k} \approx \vec{k}_0$. So we can set $\vec{k} = \vec{k}_0$ in {...}. After a little algebra,

⟹

$$\hat{\epsilon}_f \cdot \{...\} \approx -2 (\hat{\epsilon}_f \cdot \hat{\epsilon}_i) \left(\hat{k}_0 \cdot \hat{n}' \right)$$

Evaluation of the integral

In[2390]:= eq62

Out[2390]=

which gives

$$\hat{\epsilon}^* \cdot \vec{f}_{\rm sh}(\vec{k}, \vec{k}_0) \approx \frac{ik\hat{\epsilon}^* \cdot \hat{\epsilon}_0}{2\pi} \int_{\rm sh} da'\, e^{-i(\vec{k}-\vec{k}_0)\cdot\vec{x}'} (\hat{n}' \cdot \hat{k}_0). \qquad (12.62)$$

As we said, for $ka \gg 1$ the exponential is highly oscillatory, implying the integral is appreciably different from zero only for $\theta \lesssim 1/(ka)$. Under these conditions (take z' along $\hat{k}_0$)


```
In[2591]:= dap = a^2 * Sin[α]; (* dα dβ *)
npDOTk0 = Cos[α];
osc = Cos[X * Sin[α] * Cos[β]]; (* X = ka*Sin[θ] *)
Style[{"integrand=", dap * npDOTk0 * osc}, Red, 32]
ξ = Assuming[λ > 0,
   Integrate[Cos[λ * Cos[β]], {β, 0, 2 Pi}]];
ξ = ξ /. {λ → X * Sin[α]};
Style[{"I dβ = ", ξ}, Red, 32]
ξ = Integrate[Sin[α] * Cos[α] * ξ, α];
ξ = ξ /. {α → Pi / 2}
Style["failure of Mathematica", Red, 32]
```

Out[2593]= {integrand=, a^2 Cos[α] Cos[X Cos[β] Sin[α]] Sin[α]}

Out[2596]= {I dβ = , 2 π BesselJ[0, X Sin[α]]}

Out[2598]= π Hypergeometric0F1Regularized$\left[2, -\frac{X^2}{4}\right]$

Out[2599]= failure of Mathematica

The integral is $2\,\pi a^2\, J1(ka \sin\theta)/(ka \sin\theta)$.

The shadow-side contribution to the polarized cross section is

$$\frac{d\sigma}{d\Omega}_{[fi;sh]} = k^2\, a^4 \left(\frac{J1(ka \sin\theta)}{ka \sin\theta}\right)^2 \;\; |\, \epsilon_f \bullet \epsilon_i \,|^{\,2}$$

(65)

(For now we'll ignore interference between the shadow and illuminated contributions.)

The contribution to the unpolarized cross section is calculated by "sum and average" ⇒

$\frac{1}{2} \sum_{i,f} |\, \epsilon_f \bullet \epsilon_i \,|^{\,2} = \frac{1}{2} \left(1 + \cos^2\theta \right) = 1 + O\left[(ka)^{-2}\right] \approx 1$

$$\frac{d\sigma}{d\Omega}_{[un;sh]} = k^2\, a^4 \left(\frac{J1(ka \sin\theta)}{ka \sin\theta}\right)^2 \qquad (66)$$

$\theta \lesssim 1/(ka)$

The <u>*total*</u> cross section from the shadow side is

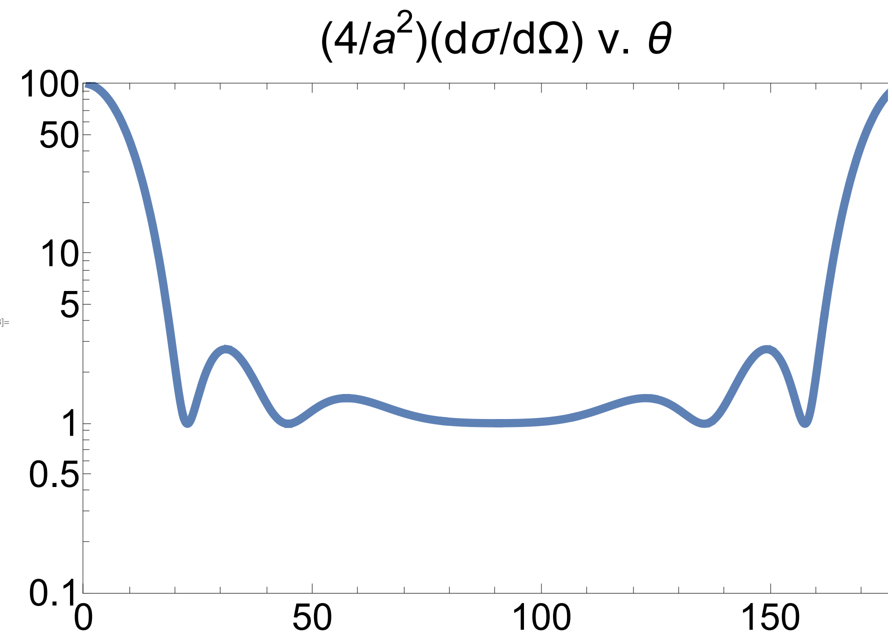
$$\sigma_{sh} = \pi\, a^2 \qquad (67)$$

Does it make sense? It seems obvious but actually it's a little surprising. See Jackson Section 10.10.

In[2647]:=
```
(* Jackson, Figure 10.16 *)
{ka = 10, a = 1}
LogPlot[1 / (a^2 / 4) * (
   a^2 / 4 +
    a^2 * (BesselJ[1, ka * Sin[θ / 180 * Pi]]) ^2 /
      (Sin[θ / 180 * Pi]) ^2),
 {θ, 0, 180}, PlotRange → {{0, 180}, {0.1, 100}},
 PlotStyle → Thickness[0.01],
 PlotLabel → "(4/a²) (dσ/dΩ)  v. θ",
 Frame → True, ImageSize → 768,
 BaseStyle → {FontSize → 32}]
```

Out[2647]= {10, 1}

Out[2648]=



In[2659]:= jackson

Out[2659]=

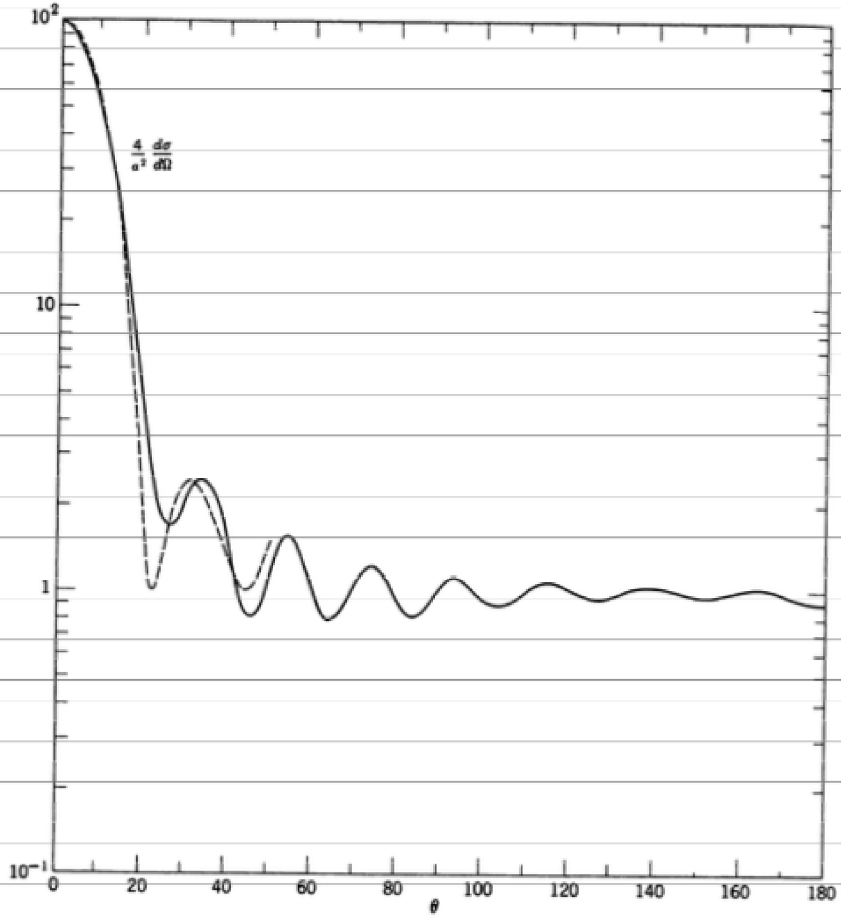


Figure 10.16 Semilogarithmic plot of the scattering cross section for a perfectly conducting sphere as a function of scattering angle, with an unpolarized plane wave incident and $ka = 10$. The solid curve is the exact result (*King and Wu*). The dashed curve is the approximation based on the sum of the amplitudes (10.127) and (10.132).

● Contribution from the *illuminated side* of S

We need to calculate this:

$$\hat{\epsilon}_f \bullet \vec{f}_{ill} =$$

$$= \frac{-ik}{4\pi} \int_{S_{ill}} da' \, e^{-i\left(\vec{k}-\vec{k_0}\right).\vec{x}'} \, \hat{\epsilon}_f \bullet$$

$$\{ -\hat{k}_0 \, (\hat{n}' \bullet \hat{\epsilon}_i) - (\hat{k}_0 - \hat{k}) \times (\hat{n}' \times \hat{\epsilon}_i) \}$$

(68)

It looks similar to f_{sh}, but it is different.

We cannot approximate $\vec{k} \approx \vec{k_0}$!

But we can use the *stationary phase approximation.*

The exponential is rapidly oscillating except where the phase is stationary as a function of $\vec{x}'$.

See WT pages 652 ~ 655.

The phase is stationary when the angle of incidence is equal to the angle of reflection, as we expect from *geometrical optics*

In[2370]:= figure12p3

Out[2370]=

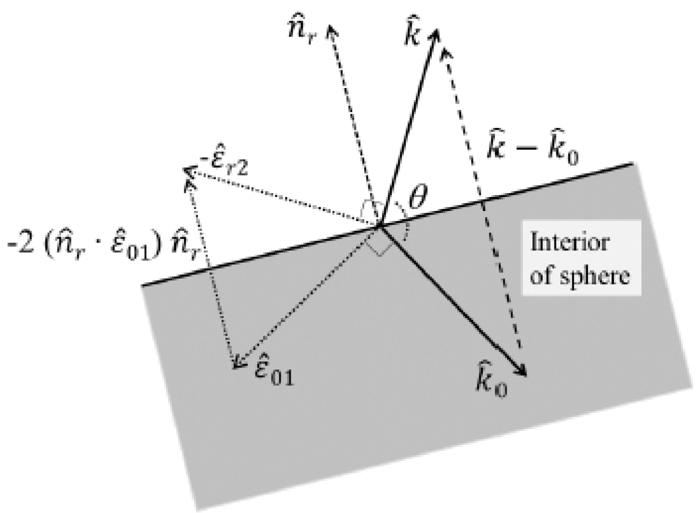


Fig. 12.3 Geometry of scattered wave (illuminated side), showing one possible polarization.

The final result is

$$\hat{\epsilon}_f \cdot \vec{f}_{\text{ill}} \approx \frac{a}{2} \exp\{-2ika \sin(\theta/2)\}\, \hat{\epsilon}_f \cdot \hat{\epsilon}_i \quad (77)$$

For unpolarized scattering, the contribution from the illuminated side is (we could have guessed this)

$$\frac{d\sigma}{d\Omega}_{[\text{un;ill}]} = \frac{a^2}{4} \quad (80)$$

$\theta \gg 1/(ka)$

and the total cross section is

$$\sigma_{\text{ill}} \approx \pi\, a^2 .$$

A Puzzle

The total cross section (σ_{sh}+ σ_{ill}) is $2\pi a^2$ = 2 × the geometrical cross section. That can't be right, can it?

Resolution of the puzzle

In[2371]:= p655

Out[2371]=

(12.81), is $2\pi a^2$. The reader might worry over this situation because this is twice the geometrical cross section. Since the cross section is just a power distribution divided by the incident flux, does this mean we extract energy from a scattering event? This nonintuitive result occurs because the incoming fields are both scattered from the illuminated surface and forward scattered to form the shadow by destructive interference with the incident wave. Thus half of the associated scattered energy is bound up in the shadow and is unavailable. There are no free lunches in physics!

Section 12.4 : The optical theorem

Homework assignment 11

Pr 11-1 = Exercise 12.1.1

Pr 11-2 = Exercise 12.3.1

Pr 11-3 = Exercise 12.3.2

Pr 11-4 = Exercise 12.3.4