12.4 The optical theorem

Theorem:

The total cross section is equal to $(4\pi/k)$ times the imaginary part of the forward scattering amplitude ;

$$\sigma_{tot} = \frac{4\pi}{k}\,\mathrm{Im}\{\hat{\epsilon}_{in} \cdot \vec{f}(\vec{k}_0, \vec{k}_0)\}$$

Example:

Recall short-wavelength scattering from a conducting sphere.

In[•]:= EQ62
EQ77

$$\hat{\epsilon}^* \cdot \vec{f}_{sh}(\vec{k}, \vec{k}_0) \approx \frac{ik\hat{\epsilon}^* \cdot \hat{\epsilon}_0}{2\pi} \int_{sh} da'\, e^{-i(\vec{k}-\vec{k}_0)\cdot\vec{x}'} (\hat{n}' \cdot \hat{k}_0). \qquad (12.62)$$

$$\hat{\epsilon}^* \cdot \vec{f}(\vec{k}, \vec{k}_0)\Big|_{ill} \approx \frac{a}{2} e^{-2ika\sin(\theta/2)} \hat{\epsilon}^* \cdot \hat{\epsilon}_{r1,2}. \qquad (12.77)$$

The forward scattering amplitude

Set $\vec{k} = \vec{k_0}$ and $\hat{\epsilon}(\vec{k}) = \hat{\epsilon}_0$

.

$$\hat{\epsilon}_0 \cdot \vec{f} = \hat{\epsilon}_0 \cdot (\vec{f}_{sh} + \vec{f}_{ill})$$

$$= \frac{ik}{2\pi} \int_{sh} da' (\hat{n}' \cdot \hat{k}_0)(\hat{\epsilon}_0 \cdot \hat{\epsilon}_0) + \frac{a}{2}(\hat{\epsilon}_0 \cdot \hat{\epsilon}_0)$$

the integral = the *projected area* = πa^2

$$= \frac{ika^2}{2} + \frac{a}{2}$$

Thus

$$\frac{4\pi}{k} \operatorname{Im} \hat{\epsilon}_0 \cdot \vec{f} = \frac{4\pi}{k} \frac{ka^2}{2} = 2\pi a^2 = \sigma_{tot}$$

Proof of the general theorem:

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In[•]:= put2;
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$\vec{E} = \vec{E}_{inc} + \vec{E}_{sc}$ and $\vec{B} = \vec{B}_{inc} + \vec{B}_{sc}$

$\vec{E}_{inc} = \epsilon_0 E_0 e^{ik0.x}$ and $\vec{B}_{inc} = \vec{k}_0 \times \vec{E}_{inc}$

■power absorbed

$$P_{abs} = -\frac{c}{8\pi} \oint_S da' \operatorname{Re}(\vec{E} \times \vec{B}^*) \cdot \hat{n}'$$

■power scattered

$$P_{sc} = \frac{c}{8\pi} \oint_S da' \operatorname{Re}(\vec{E}_{sc} \times \vec{B}_{sc}^*) \cdot \hat{n}'$$

■incident power

$P_{inc} = \frac{c}{8\pi}\oint_S da' \, Re(\vec{E}_{inc} \times \vec{B}_{inc}^*) \cdot \hat{n}'$

$P_{inc} = 0$

■extinguished power $P_{ext}=P_{abs}+P_{sc}$

$P_{ext} = -\frac{c}{8\pi}\oint_S da'$

▫ $Re(\vec{E}_{sc} \times \vec{B}_{inc}^* + \vec{E}_{inc}^* \times \vec{B}_{sc}) \cdot \hat{n}'$

.

$P_{ext} = -\frac{c}{8\pi} Re\{ E_0^* \oint_S da' \exp(-ik_0.x')$

$[\hat{n}'.(\vec{E}_{sc}\times(\vec{k}_0 \times \vec{\epsilon}_0))+ \hat{n}'.(\hat{\epsilon}_0\times \vec{B}_{sc})]\}$

Some vector algebra $\Longrightarrow$

$P_{ext} = -\frac{c}{8\pi} Re\{ E_0^* \oint_S da' \exp(-ik_0.x')$

$\epsilon_0.[\hat{n}'\times \vec{B}_{sc} + \hat{k}_0 \times (\hat{n}' \times \vec{E}_{sc})]$

Now go back and look up the general equation for the scattering amplitude

In[◦]:= EQ49

Out[◦]=

$$\hat{\epsilon}^* \cdot \vec{f}(\vec{k},\vec{k}_0) = \frac{ik}{4\pi E_0}\oint_S da' \, e^{-i\vec{k}\cdot\vec{x}'}\left[\hat{\epsilon}^* \cdot (\hat{n}' \times \vec{B}_{sc}) + \hat{\epsilon}^* \cdot (\hat{k} \times (\hat{n}' \times \vec{E}_{sc}))\right]. \quad (12.49)$$

Thus,

$$P_{ext} = \frac{c}{2k} \text{Im}\{ E_0^* E_0 \hat{e}_0 \cdot \vec{f}(\vec{k}_0, \vec{k}_0)]$$

The result is

$$\sigma_{tot} = \frac{Pext}{inc.flux} \quad \text{where inc.flux} = \frac{c}{8\pi} \, |E_0|^2$$

.

$$\sigma_{tot} = \frac{4\pi}{k} \text{Im}\{ \hat{\epsilon}_{in} \cdot \vec{f}(\vec{k}_0, \vec{k}_0)\}$$