1-

Jackson 10.1
Scattering at long wavelengths

$d \ll \lambda$

The incident $\vec{E}$ field induces an oscillating current in the scatterer ; $\Longrightarrow$ radiation, which is the scattered wave.

(A) Scattering by a small induced dipole

Because $d \ll \lambda$, the electric dipole moment will dominate the radiation.

Time dependence exp[-i ω t] is understood.

$\vec{E}_{inc}(\vec{x}) = \hat{\epsilon}_0 E_0 e^{i \vec{k}_0 \cdot \vec{x}}$

$\vec{B}_{inc} = \hat{k}_0 \times \vec{E}_{inc}$

$\vec{E}_{sc} = k^2 \frac{e^{ikr}}{r} \{ \hat{n} \times (\vec{p} \times \hat{n}) - \hat{n} \times \vec{m} \}$ /far zone/

$\vec{B}_{sc} = \hat{n} \times \vec{E}_{sc}$

Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{E_0^2} | \hat{\epsilon}_f \cdot \vec{p} + (\hat{n} \times \hat{\epsilon}_f) \cdot \vec{m} |^2$$

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{E_0^2} | \hat{\epsilon}_f \cdot \vec{p} + (\hat{n} \times \hat{\epsilon}_f) \cdot \vec{m} |^2$$

- Remember, dipoles $\vec{p}$ and $\vec{m}$ are *induced* dipoles; the moments depend on $\vec{E}_{inc}$ and $\vec{B}_{inc}$; i.e., they depend on E_0 and $\hat{\epsilon}_0$.
- $k = k_0 = \omega/c$. So the cross section is proportional to ω^4. (Jackson calls this dependence "Rayleigh's law". Don't confuse this with Rayleigh scattering — scattering of light by a bound electron.)

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(B) Scattering by a small dielectric sphere

As a simple example, consider a sphere with radius a and electric permittivity $\epsilon(\omega)$. Approximate $\epsilon(\omega)$ by the static permittivity, $\epsilon = \epsilon(0)$.

The electric dipole moment (from electrostatics; related to the Clausius-Mossotti relation)

$$\vec{p} = \frac{\epsilon-1}{\epsilon+2} a^3 \vec{E}_{inc}$$

$\vec{m} = 0$

$\Longrightarrow$ the polarized cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{f,i} = k^4 a^6 \left| \frac{\epsilon-1}{\epsilon+2} \right|^2 (\hat{\epsilon}_f \cdot \hat{\epsilon}_i)^2$$

Unpolarized scattering (see Jackson)

- Analyze the polarization and propagation vectors
- Sum and average $\Longrightarrow$

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$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = k^4\, a^6 \left|\frac{\epsilon-1}{\epsilon+2}\right|^2 \cdot \frac{1}{2}\left(1+\cos^2\theta\right)$$

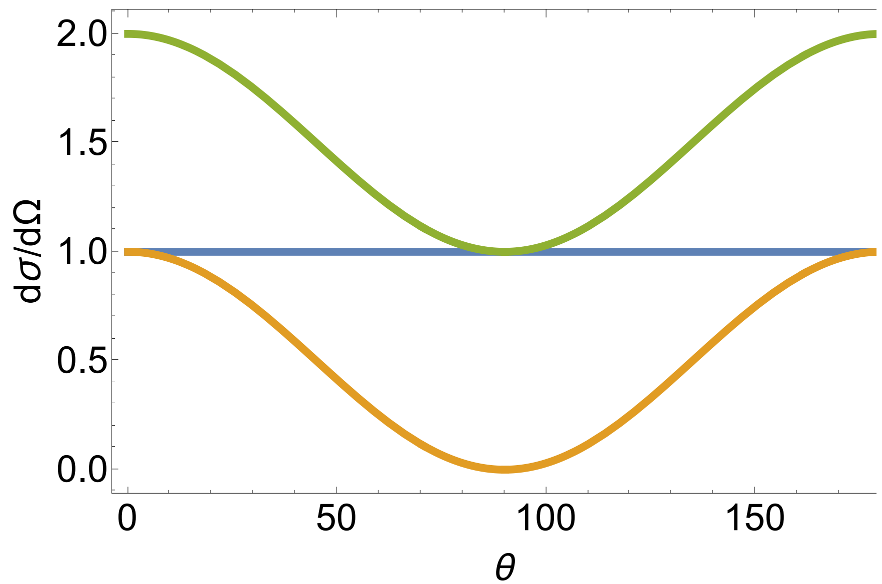
↖ scattered wave polarizations:
perpendicular ⇒ isotropic term
parallel ⇒ $\cos^2\theta$ term
(i.e., perp. and para. to the scattering plane)

$$\sigma_{\text{total}} = \frac{8\pi}{3}\, k^4\, a^6 \left|\frac{\epsilon-1}{\epsilon+2}\right|^2$$

Graphical analysis

In[60]:=
```
Plot[{1, Cos[θ*Pi/180]^2, 1+Cos[θ*Pi/180]^2}, {θ, 0, 180},
 Frame → True, FrameLabel → {"θ", "dσ/dΩ"}, ImageSize → 768,
 PlotStyle → Thickness[0.01],
 BaseStyle → {32}]
```

Out[60]=



The scattered radiation is 100% polarized at θ=90°.

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Try it yourself: Sky light is polarized in the direction 90° from the sun.

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(C) Scattering by a small conducting sphere

Consider a small metal sphere with radius a;

e.m. wave scattering with $\lambda \gg a$,

e.g. microwaves.

$\vec{p} = a^3 \vec{E}_{inc}$ and $\vec{m} = -2\pi a^3 \vec{B}_{inc}$

Exercise; electro and magneto statics

Jackson shows

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} k^4 a^6 \cdot \begin{cases} [\cos\theta - 1/2]^2 & \text{para} \\ [1 - (1/2)\cos\theta]\hat{}\,2 & \text{perp} \end{cases}$$

⟸ unpolarized incident waves;

parallel and perpendicular polarizations of the scattered waves;

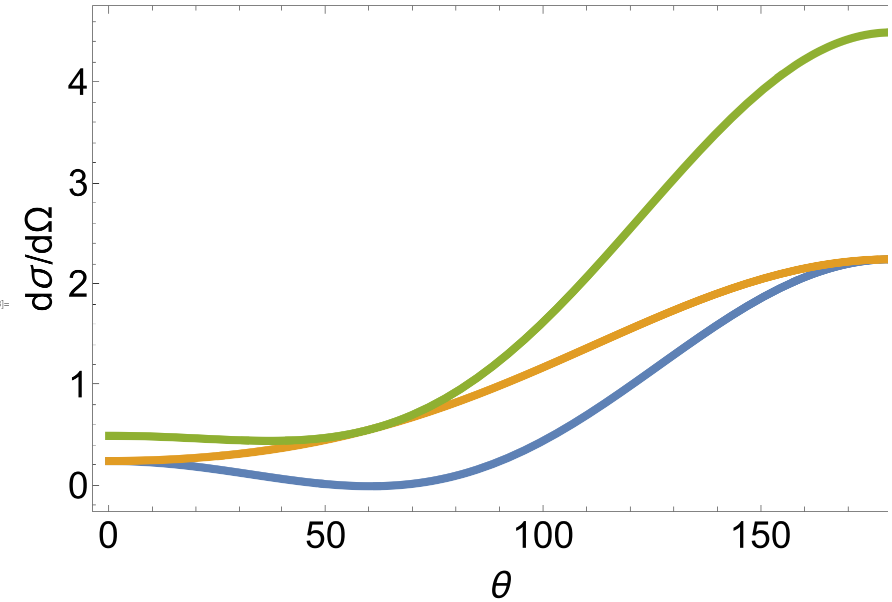
i.e. para and perp to the plane of scattering.

Graphical analysis:

In[113]:=
```
Remove[fa, fp, θ, α];
fa[α_] := (Cos[θ] - 1 / 2) ^2 /. {θ → α / 180 * Pi}
fp[α_] := (1 - 1 / 2 * Cos[θ]) ^2 /. {θ → α / 180 * Pi}
```

In[118]:=
```
Plot[{ fa[α], fp[α], fa[α] + fp[α]} , {α, 0, 180},
 Frame → True, FrameLabel → {"θ", "dσ/dΩ"}, ImageSize → 768,
 PlotStyle → Thickness[0.01],
 BaseStyle → {32}]
```

Out[118]=



The strong backward peaking ($\theta \approx 180$) is due to interference between the electric dipole amplitude and the magnetic dipole amplitude.

"Whole books are devoted to the scattering of light by spherical particles with arbitrary μ, ϵ, σ". - Jackson

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(D) Collection of scatterers

If the scatterers are located at the points of a crystal lattice, it describes *Bragg scattering* — Xray scattering from a crystal.