1-

Jackson, Section 10.5
Scalar Diffraction Theory

In[•]:= `fig10p7`

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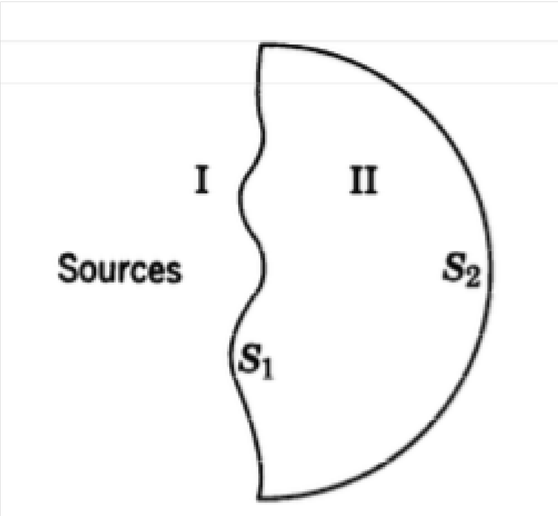

In a diffraction problem there are two spatial regions:

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I: the source region, where radiation occurs;

II: the diffraction region.

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Also, there are two boundary surfaces:

S_1 = “the screen”; an opaque surface with apertures;

S_2 = a surface "at infinity".

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Let $\psi(\vec{x},t)$ be a scalar field, with

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(\vec{x},t) = \sigma(\vec{x},t) \text{ (= sources)}$$

Suppose the sources and $\psi(\vec{x},t)$ are harmonic in time, with frequency $\frac{\omega}{2\pi}$.

$$\psi(\vec{x},t) = \psi(\vec{x})\, e^{-i\omega t}$$

In the diffraction region (Region II)

$(\nabla^2 + k^2)\,\psi(x) = 0$ where $k = \omega/c$.

= the Helmholtz wave equation

The Green' s function $G(\vec{x}, \vec{x}\,')$

$G(\vec{x}, \vec{x}\,')$ is partially defined by

$$(\nabla^2 + k^2)\, G(\vec{x}, \vec{x}\,') = -\delta^3(\vec{x} - \vec{x}\,')$$

(Jackson's convention)

We'll also need boundary conditions, later.

Recall the second Green identity,

$$\int_V d^3x\, \left(\psi\,\nabla^2\phi - \phi\,\nabla^2\psi\right) =$$

$$= \oint_S da\,(\psi\,\hat{n}\cdot\nabla\phi - \phi\,\hat{n}\cdot\nabla\psi)$$

Let $\phi = G$ and $\psi = \psi \Longrightarrow$

$$\psi(\vec{x}) = \oint_S da'\left[\psi(\vec{x}\,')\,\hat{n}'\cdot\nabla' G - G\,\hat{n}'\cdot\nabla'\psi(\vec{x}\,')\right]$$

3-

The Kirchoff diffraction integral

Take $G(\vec{x}, \vec{x}')$ be the free space Green's function for outgoing waves

corresponds to certain boundary conditions

$$G(\vec{x}, \vec{x}') = \frac{e^{ikR}}{4\pi R} \text{ where } R = |\vec{x} - \vec{x}'|$$

Jackson's convention

$\Longrightarrow$

$$\psi(\vec{x}) = \oint_S da' \frac{e^{ikR}}{4\pi R} \hat{n}' \bullet \left[-\nabla' \psi(\vec{x}') - ik\left(1 + \frac{i}{kR}\right) \frac{\vec{R}}{R} \psi(\vec{x}') \right]$$

- Now, we want V = Region II and S = $S_1 + S_2$

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- On S_2 (the boundary surface at infinity)

$$\psi(\vec{x}') \longrightarrow \frac{e^{ikr}}{r} f(\theta,\phi) \quad \sim O(1/r)$$

$$\nabla' \psi(\vec{x}') \quad \sim O(1/r)$$

$\oint_{S_2} (\ldots) \sim O(1/r)$ *(exercise)* $\longrightarrow 0$.

- Thus,

$$\psi(\vec{x}) = \int_{S_1} da' \frac{e^{ikR}}{4\pi R} \hat{n}' \bullet \left[-\nabla' \psi(\vec{x}') - ik\left(1 + \frac{i}{kR}\right) \frac{\vec{R}}{R} \psi(\vec{x}') \right]$$

(10.79)

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(10.79)

$$\psi(\vec{x}) = \int_{S_1} da' \, \frac{e^{ikR}}{4\pi R} \, \hat{n}' \cdot \left[-\nabla' \psi(\vec{x}') - ik\left(1 + \frac{i}{kR}\right) \frac{\vec{R}}{R} \psi(\vec{x}') \right]$$

Eq. (10.79) is the *Kirchhoff integral formula* for a scalar field.

Here is what it means:

The field throughout the diffraction region (= Region II) is determined by the field on the diffracting screen (= S_1).

Now the question is, how can we use Eq. (10.79)?

Kirchhoff's approximation

(1) $\psi = 0$ and $\partial\psi/\partial n = 0$ for all points on S_1, except in the apertures.

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(2) In the apertures of S_1, ψ and $\partial\psi/\partial n$ are equal to the values of the incident plane wave if there were no screen.

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Jackson states that “the standard diffraction calculations of classical optics are all based on the Kirchhoff approximation.”

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But Jackson points out that “the approximations (1) and (2) are mathematically inconsistent”; and “(10.79) does not yield the assumed values of ψ and $\partial\psi/\partial n$ in the apertures.

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The Kirchhoff approximation may provide a qualitatively correct picture of diffraction, but for an accurate theory we need something better.

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Furthermore, this is only *scalar diffraction theory*. For light and other E.M. waves we need a vector extension of Kirchhoff's integral formula.

Further developments

Eq (10.79) can be improved by using a better Green's function.

For example, let $G_D(\vec{x}, \vec{x}')$ be the Green's function for *Dirichlet boundary conditions*; i.e.,

$$G_D(\vec{x}, \vec{x}') = 0 \text{ for } \vec{x}' \in S .$$

Use G_D instead of the free Green's function G. Then (10.79) is replaced by

$$\psi(\vec{x}) = \int_{S_1} da' \;\; \psi(\vec{x}') \;\; \hat{n}'\cdot\nabla' G_D(\vec{x}, \vec{x}') \qquad (10.81)$$

Now we can *consistently* assume that $\psi = 0$ on S_1 except in the apertures, and ψ = the incident wave in the apertures.

Alternatively,

we could use the Green's function $G_N(\vec{x}, \vec{x}')$ for *Neumann boundary conditions,* and approximate $\partial\psi/\partial n'$ on the screen.

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A special case, relevant to Fraunhofer diffraction.

Suppose the diffraction screen (= S_1) is an *infinite plane* (with apertures).

Then, using the Dirichlet boundary conditions (and ψ approximated on S_1),

$$\psi(\vec{x}) = \frac{k}{2\pi i} \int_{S_1} da' \frac{e^{ikR}}{R} \left(1 + \frac{i}{kR}\right) \frac{\hat{n}'\cdot\vec{R}}{R} \psi(\vec{x}')$$

The Fresnel Huygens Integral

Finally, Jackson's Eq (10.86):

$$\psi(P) = \frac{k}{2\pi i} \int_{\text{apertures}} da' \frac{e^{ikr}}{r} \frac{e^{ikr'}}{r'} \text{Ob}(\theta,\theta')$$

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where Ob is the obliquity factor

$$\text{Ob} = \begin{cases} \cos\theta & \text{Dirichlet } b.c. \\ \cos\theta' & \text{Neumann } b.c. \end{cases}$$

Quiz

The quiz is a question that every physicist should be able to answer.

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In words and pictures,
  describe and explain ...
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In[7]:=

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