Describe and explain the Arago spot.

(1) It is the tiny bright spot in the shadow of a smooth sphere. (2) It is an example of Fresnel diffraction, occurring in the near zone. (3) It come from constructive interference of Huygens wavelets emitted around the projected circumference of the sphere. (4) It was the *Experimentum Crucis* for the wave theory of light.

Problem 11-5. From the Oxford Dictionary of Physics...

Fraunhofer diffraction: the light sourer and receiving screen are in effect at infinite distances from the diffracting object, so that wave fronts are planar. (2 points)

Fresnel diffraction: the light source or receiving screen or both are at finite distances from the diffracting object, so that wave fronts are not planar.

1-

Scalar diffraction examples

Review

Last time we derived the general equations for scalar diffraction theory. (Jackson, Section 10.9 AND Wilcox-Thron Section 12.6)

∎ To do a scalar diffraction calculation we have several choices:

- Kirchhoff approximation; $G_{free}(x,x') = e^{ikR}/R$
- Dirichlet boundary condition; $G_D(x,x')$
- Neumann boundary condition; $G_N(x,x')$

∎ The usual "song and dance" gives

$$\Phi(\vec{x}) = \frac{1}{4\pi}\oint da' \left[\partial_n G\left(\vec{x}, \vec{x}'\right) \Phi(\vec{x}') - \partial_n \Phi(\vec{x}')\, G(\vec{x}, \vec{x}') \right] \tag{12.155}$$

2nd term = 0 for Dirichlet b.c

1st term = 0 for Neumann b.c.

∎The surface integral reduces to integration over the aperture(s) where we approximate either Φ or $\partial_n \Phi$ by the incident wave.

In the Fraunhofer approximation,

$$\Phi(\vec{x}) = \frac{ik\,F\,e^{ikr}}{2\pi r}\,\Phi_0 \int_A da' \exp\{-i\,(\vec{k}-\vec{k}_0)\cdot\vec{x}'\} \tag{12.165}$$

where $F_D = \cos\theta$ and $F_N = -\cos\alpha = -\hat{n}'\cdot\vec{k}_0$.

Scalar diffraction examples (WT Section 12.7)

Example 1: Plane waves incident on a circular aperture at non-normal incidence

Start with eq (12.165), and use cylindrical coordinates (ρ' , ϕ', z').

$\vec{x}' = \{ \rho' \cos\phi' , \ \rho' \sin\phi', z' \}$

at z' = 0 in the aperture

$\vec{k_0} = k \{ \sin\alpha, 0, \cos\alpha \}$;

α = the angle of incidence

$\vec{k} = k \{ \sin\theta \cos\phi , \sin\theta \sin\phi , \cos\theta \}$

Draw a picture.

$(\vec{k} - \vec{k_0}) \cdot \vec{x}' =$

$= \ k \rho' (\sin\theta \cos(\phi-\phi') - \sin\alpha \cos\phi')$

Define Δ and δ by

$\Delta \cos(\phi' + \delta) \equiv \sin\theta \cos(\phi-\phi') - \sin\alpha \cos\phi'$

Solve for Δ

$\Delta^2 = \sin^2\theta + \sin^2\alpha - 2 \sin\alpha \sin\theta \cos\phi$

we won't need δ

Exercise   *Evaluate* $\int_0^a \int_0^{2\pi} d\rho' \, \rho' \, d\phi' \, (..) \Longrightarrow$

$$\int_A da' \exp[-i\,(\vec{k}-\vec{k_0})\cdot\vec{x}'\,] = 2\pi\,\frac{a}{k}\,\frac{J_1(ka\,\Delta)}{\Delta}$$

The field in the diffraction region is

$$\Phi(\vec{x}) = \frac{i\,F\,a\,e^{ikr}}{r}\,\Phi_0\,\frac{J_1(ka\Delta)}{\Delta}$$

and the "*differential transition rate*" is

$$\frac{dT}{d\Omega} = \frac{dP_{out}}{P_{in}\,d\Omega}$$

$$\frac{dT}{d\Omega} = \frac{F^2}{\pi\cos\alpha}\left(\frac{J_1(ka\Delta)}{\Delta}\right)^2$$

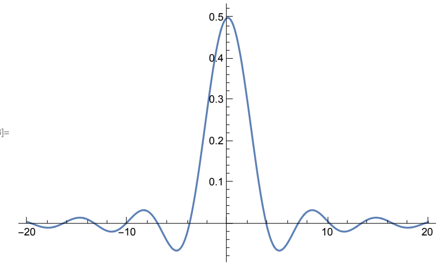
see Exercise 13.10.4;
what is the scalar 'Poynting vector'?

Maximum amplitude

The waves are coming in at angle of incidence α.
In what direction $(\theta\,,\,\phi)$ is the maximum intensity?
The max occurs at $\Delta =$
0. The incident wave has $\phi = 0$,
i.e., along the x axis; then $\Delta^2 = (\sin\theta - \sin\alpha)^2$.
$\Delta = 0 \Longrightarrow \theta = \alpha$, which agrees with geometrical optics.

In[3]:=
```
Plot[BesselJ[1, x] / x,
 {x, -20, 20}, PlotRange → {All, All}]
```

Out[3]=



Example 2 : Diffraction from a long slit of width d, at normal incidence

Same figure as the quiz.

Start here

$$\Phi(\vec{x}) = \frac{ik F e^{ikr}}{2\pi\rho} \Phi_0 \int_{slit} da' \exp[-i \vec{k} \cdot \vec{x}' + ik\rho'^2/(2\rho)] . \qquad (12.180)$$

where

$$F = \begin{cases} \cos\phi & \text{Dirichlet } b.c. \\ -1 & \text{Neumann } b.c. \end{cases}$$

Derive (12.180)

$$G(\vec{x}, \vec{x}') = \frac{\exp\{ik|\vec{x}-\vec{x}'|\}}{|\vec{x}-\vec{x}'|}$$

$$|\vec{x}-\vec{x}'| \approx r - \hat{n}\cdot\vec{x}' + \frac{(\rho')^2}{2\rho} \quad \text{(far zone)}$$

Usually we drop the third term but now we will keep it for greater accuracy.

Cylindrical coordinates: $\vec{x} = \rho\,\{\sin\psi, 0, \cos\psi\}$;

$\hat{k} = \hat{n}\,\{\sin\psi, 0, \cos\psi\}$

i.e., observation occurs in the xz-plane.

Incident waves are normal, so $\hat{k}_0 = \hat{e}_z = \{0,0,1\}$;

$\therefore e^{i\vec{k}_0 \cdot \vec{x}'} = 1$ in the slit.

The integral $\iint$ dx’ dy’ [..]

slit in the xy-plane

$[x'] = d$; $[y'] = \infty$; $z' = 0$

$$\int_{\text{slit}} da' \exp[-i\, \vec{k} \cdot \vec{x}' + ik\, \rho'^2 / (2\rho')] = \text{Iy Ix}$$

$$\text{Iy} \equiv \int_{-\infty}^{\infty} dy' \exp[ik\, y'^2/(2\rho)]$$

$$\text{Ix} \equiv \int_{-d/2}^{d/2} dx' \exp[ik\, x'^2/(2\rho) - i\, kx' \sin\psi]$$

$$\vec{x}' = \{x', y', 0\}$$

$$\vec{k} = k\{\sin\psi, 0, \cos\psi\}$$

We’ll need the *Fresnel Integrals,* defined

$$C(u) \equiv \int_0^u dt \cos(\pi t^2/2)$$

$$S(u) \equiv \int_0^u dt \sin(\pi t^2/2)$$

In[=]:=

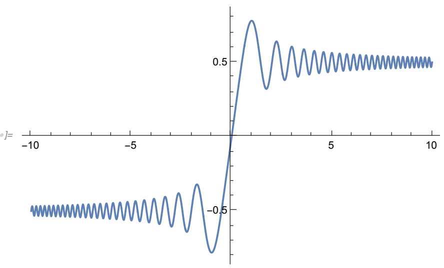
```
? FresnelC
? FresnelS
```

FresnelC[z] gives the Fresnel integral C(z). >>

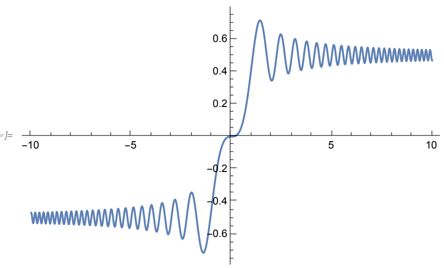
FresnelS[z] gives the Fresnel integral S(z). >>

```
In[*]:= Plot[FresnelC[u], {u, -10, 10}]
Plot[FresnelS[u], {u, -10, 10}]
```

Out[*]=



Out[*]=



Properties:

Both C(u) and S(u) are odd functions of u.

$C(\infty) = S(\infty) = \frac{1}{2}$

Integral Iy

$$\mathrm{Iy} = \mathrm{Iy} = \int_{-\infty}^{\infty} dy' \exp[i\, k\, y'^2/(2\rho)]$$

$$= \int_{-\infty}^{\infty} dy' \; \{ \cos(\lambda y'^2) + i \sin(\lambda y'^2) \; \}$$

$$\lambda = k/(2\rho)$$

$$\lambda\, y'^2 = \pi\, t^2/2$$

$$= 2\sqrt{\frac{\pi}{2\lambda}} \; [\; C(\infty) + i\, S(\infty) \;] = 2\sqrt{\frac{\pi\,(2\rho)}{2k}} \;\; \frac{1}{2}\,(1+i)$$

$$= \sqrt{\frac{\pi\,\rho}{k}}\;(1+i)$$

Integral Ix

$$Ix = \int_{-d/2}^{d/2} dx' \exp[ik(x')^2/(2\rho) - i\, k\, x' \sin\psi]$$

$$= \int_{-d/2}^{d/2} dx' \exp[\, ik/(2\rho)\, (x'^2 - 2\, x' \rho \sin\psi)\,]$$

$$= \int_{-d/2}^{d/2} dx' \exp[\, ik/(2\rho)\, (x' - \rho \sin\psi)^2\,]$$

$$\cdot \exp[-i\,(k\rho/2) \sin^2\psi\,]$$

$$= \exp[-i\,(k\rho/2) \sin^2\psi] \sqrt{\frac{\pi\rho}{k}}$$

$$\cdot \{\, C(a-b) + iS(a-b) - C(-a-b) - iS(-a-b)\,\}$$

where

$$a = \sqrt{\frac{kd^2}{4\pi\rho}} \quad \text{and} \quad b = \sqrt{\frac{k\rho}{\pi}} \sin\psi = \sqrt{\frac{k x^2}{\pi\rho}}$$

N.B. $x = \rho \sin\psi$

Define the Fresnel number, $N_F \equiv \dfrac{kd^2}{8\pi\rho}$ (*small !*)

$a = \sqrt{2 N_F}$ and $b = \sqrt{2 N_F}\,(2x/d)$

The "differential transition rate" is defined by

$$\frac{dT}{d\psi} = \frac{\rho}{d} \frac{|\Phi(\vec{x})|^2}{|\Phi_0|^2} = \frac{\rho}{d} \left(\frac{kF}{2\pi\rho}\right)^2 Iy^2\, Ix^2$$

as a function of ρ and ψ; $x = \rho \sin\psi$.

Interesting exercise: plot $dT/d\Omega$ for various values of ρ, d, λ.

The Fraunhofer limit

Take the limit $\rho \longrightarrow \infty$.

As u $\longrightarrow \pm \infty$,

$C(u) \approx \frac{1}{2}\,\text{sign}(u) + \frac{1}{\pi u}\sin\left(\frac{\pi}{2}\,u^2\right)$

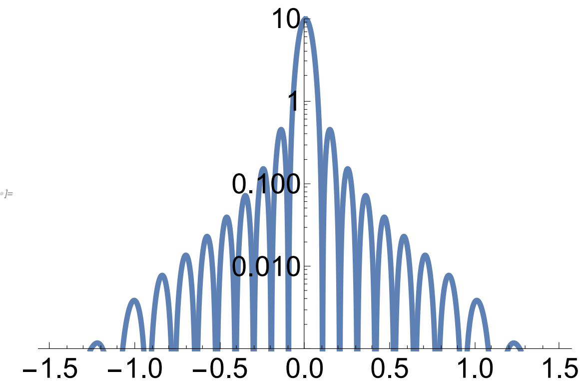
$S(u) \approx \frac{1}{2}\,\text{sign}(u) - \frac{1}{\pi u}\cos\left(\frac{\pi}{2}\,u^2\right)$

Exercise 12.7.3 $\Longrightarrow$ “beautiful and simple Fraunhofer slit diffraction formula”

$$\frac{dT}{d\psi} = \frac{2\,F^2}{\pi\,kd}\left(\frac{\sin\left(\frac{kd}{2}\sin\psi\right)}{\sin\psi}\right)^2$$


```
(* plot *)
(*Remove["Global`*"]
*)F[ψ_] = {Cos[ψ], -1}[[1]];
d = 1; λ = 0.1; k = 2 π / λ; k * d
dT[ψ_] = 2 F[ψ] ^ 2 / (π * k * d) *
   Power[Sin[k * d / 2 * Sin[ψ]] / Sin[ψ], 2];
plt[1] = LogPlot[dT[ψ], {ψ, -Pi / 2, Pi / 2},
  PlotRange → {{-Pi / 2, Pi / 2}, {0.001, 10}},
  PlotStyle → Thickness[0.01],
  BaseStyle → {24}, ImageSize → 480]
```

```
62.8319
```



Nodes occur at kd sin $\psi_n = 2\pi$ n ;   n∊Z

Compare Dirichlet and Neumann assumptions ...