Vector diffraction examples

Wilcox and Thron, Section 12.8

We've learned some things about diffraction from the scalar theory. Still, that is not electromagnetism. So now we'll extend the theory to electromagnetic waves. We can use some of the general ideas and techniques again, but here we will need to include polarizations.

Recall

- Scalar $\Phi(\vec{x},t)$; $(\nabla^2+k^2)\,\Phi = 0.$

$$\Phi(\vec{x}) = \frac{1}{4\pi}\int_S \mathrm{da}'\,[\,\partial_n G\left(\vec{x},\,\vec{x}'\right)\,\Phi(\vec{x}') - \partial_n \Phi(\vec{x}')\,G(\vec{x},\,\vec{x}')\,]$$

2nd term = 0 for Dirichlet G.F.; specify Φ on S;

1st term = 0 for Neumann G.F.; specify $\partial_{n'}\Phi$ on S.

Consider a planar screen (= the xy-plane) and incident e.m. waves in the direction $\hat{k}_0$.

- We can assume $A_z = 0$ at S because any currents are in the screen (= xy-plane). ∴ use Neumann boundary conditions,

$$\vec{A}(\vec{x}) = -\frac{1}{2\pi}\int_S \mathrm{da}'\;\partial_{n'}\vec{A}(\vec{x}')\;G\left(\vec{x},\,\vec{x}'\right) \quad (12.160)$$

$$= \frac{1}{2\pi}\int_S \mathrm{da}'\;\hat{n}'\times\vec{B}(\vec{x}')\;G\left(\vec{x},\,\vec{x}'\right)$$

$$\vec{B}(\vec{x}) = \frac{1}{2\pi}\nabla\times\int_S \mathrm{da}'\;\hat{n}'\times\vec{B}(\vec{x}')\;G\left(\vec{x},\,\vec{x}'\right)$$

∎ Duality transformation

In free space,

$$\nabla \times \vec{B} = -\frac{1}{c}\,\frac{\partial \vec{E}}{\partial t} \quad \text{and} \quad \nabla \times \vec{E} = \frac{1}{c}\,\frac{\partial \vec{B}}{\partial t}\,.$$

Let $\vec{E} \longrightarrow \vec{E}_d \equiv \vec{B}$ and $\vec{B} \longrightarrow \vec{B}_d \equiv -\vec{E}$;

then

$$\nabla \times \vec{B}_d = -\frac{1}{c}\,\frac{\partial \vec{E}_d}{\partial t} \quad \text{and} \quad \nabla \times \vec{E}_d = \frac{1}{c}\,\frac{\partial \vec{B}_d}{\partial t}\,.$$

So substitute $\vec{B}_d$ into the equation $\Longrightarrow$

$$\vec{E}(\vec{x}) = \frac{1}{2\pi}\,\nabla \times \int_S da'\ \hat{n}' \times \vec{E}(\vec{x}')\ G\left(\vec{x}, \vec{x}'\right)$$

(12.200)

(WT give a second derivation)

$$\vec{E}(\vec{x}) = \frac{1}{2\pi}\,\nabla \times \int_S da'\ \hat{n}' \times \vec{E}(\vec{x}')\ G\left(\vec{x}, \vec{x}'\right)$$

Result

The diffracted field $\vec{E}(\vec{x})$ in the diffraction volume is determined by the field $\vec{E}(\vec{x}')$ on the screen.

Now suppose the screen is a perfect conductor with apertures.

The obvious approximation for $\hat{n}' \times \vec{E}(\vec{x}')$ is ...

- $\hat{n}' \times \vec{E}(\vec{x}') = 0$ on the barrier region of S
- $\hat{n}' \times \vec{E}(\vec{x}') =$ the incident wave in the aperture

(Huygens principle)

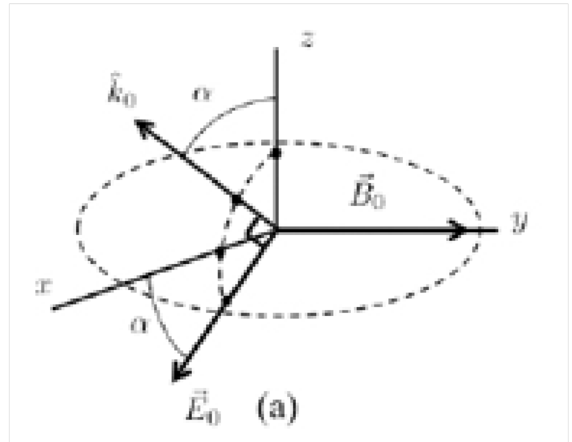
Example: Electromagnet diffraction for a circular aperture at non-normal incidence

$$\vec{E}(\vec{x}) = \frac{1}{2\pi} \nabla \times \int_S da' \; \hat{n}' \times \vec{E}(\vec{x}') \;\; G\left(\vec{x}, \vec{x}'\right)$$

- Aperture is in the xy-plane; radius of the aperture = a ;
- $\vec{k}_0$ is in the xz-plane, at angle α from the z axis ;
- Separate two plane polarizations (see Figure 12.5)

$B_\perp$ polarization

(a) transverse magnetic ($B_\perp$) polarization; $\vec{B}$ is perpendicular to the xz-plane; $B_0\, \hat{e}_y$; the electric field is in the xz-plane at an angle α with respect to the screen = angle $\alpha + \frac{\pi}{2}$ w.r.t the z axis.



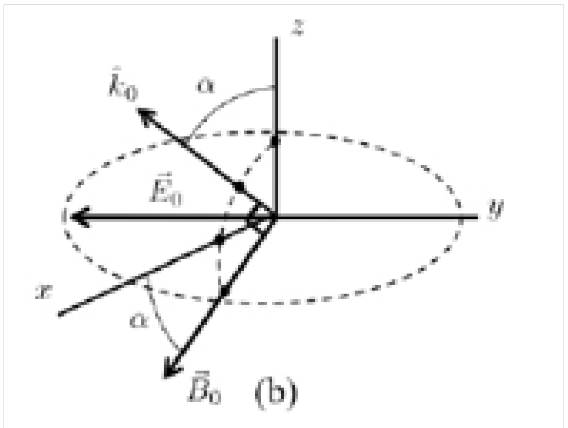
$$\vec{E}(\vec{x}) = \frac{1}{2\pi} \nabla \times \int_S da' \; \hat{n}' \times \vec{E}(\vec{x}') \;\; G\left(\vec{x}, \vec{x}'\right)$$

$$ik\, \hat{n} \times (\hat{e}_z \times (\hat{e}_x\, c_\alpha - \hat{e}_z s_\alpha))\, E_0 \, \frac{e^{ikr}}{r}\; e^{-ik\, \hat{n} \cdot \vec{x}'}$$

$ik\, \hat{n} \times \hat{e}_y \cos\alpha$ ▫ scalar terms

$E_\perp$ polarization

(b) transverse electric ($E_\perp$) polarization; the electric field is
$-E_0\,\hat{e}_y$; the magnetic field is at angle $\alpha + \frac{\pi}{2}$ w.r.t. the z axis.



$$\vec{E}(\vec{x}) = \frac{1}{2\pi} \nabla \times \int_S da' \;\; \hat{n}' \times \vec{E}(\vec{x}') \;\; G\left(\vec{x}, \vec{x}'\right)$$

$$ik\,\hat{n} \times (\hat{e}_z \times (-\hat{e}_y))\, E_0\, \frac{e^{ikr}}{r}\, e^{-ik\,\hat{n}\cdot\vec{x}'}$$

$$ik\,\hat{n} \times \hat{e}_x \;\square\; \text{scalar terms}$$

Result is equation (12.213)

$$\vec{E}(\vec{x}) = \frac{i\,\vec{F}\,a}{r}\, E_0\, \frac{J_1(ka\,\Delta)}{\Delta}$$

where

$$\Delta^2 = \sin^2\theta + \sin^2\alpha - 2\sin\alpha\,\sin\theta\,\cos\phi$$

and

$$\vec{F} = \begin{cases} \cos\alpha\,(\hat{n}\times\hat{e}_y) & [B_\perp \text{ pol.}] \\ n\times\hat{e}_x & [E_\perp \text{ pol.}] \end{cases}$$

Transition coefficient

$$P_{\text{in}} = \frac{c}{8\pi} E_0^2 \text{ A } \cos\alpha$$

$$\frac{\text{dP}_{\text{out}}}{\text{d}\Omega} = r^2 \frac{c}{8\pi} \left| \vec{E}(\vec{x}) \right|^2$$

$$\frac{\text{dT}}{\text{d}\Omega} = \frac{r^2}{A\cos\alpha} \left| \frac{\vec{E}(\vec{x})}{E_0} \right|^2$$

$$= \frac{\vec{F}^2}{\pi\cos\alpha} \left| \frac{J_1(\text{ka}\Delta)}{\Delta} \right|^2$$

which you need for a homework problem.