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Cherenkov Radiation : The Frank-Tamm formula

Jackson, Sections 13.3.and 13.4

A charged particle moves with constant velocity in a dielectric material. Cherenkov radiation is emitted if the speed of the particle is greater than the speed of light.

Now,calculate the intensity ≡ the energy flux.

$\frac{d^2 E}{dx\, d\omega}$ = radiated energy per unit length of the path, per unit frequency

Parameters

charge = z e

velocity = $\vec{v} = v\, \hat{e}_x$; also $\beta = v/c$

$\epsilon(\omega)$ = permittivity

$n(\omega) = \sqrt{\epsilon(\omega)}$ = index of refraction.

The Frank-Tamm formula (in Gaussian units)

$$\frac{d^2 E}{dx\, d\omega} = \frac{z^2 e^2}{c^2}\ \omega\ \left(1 - \frac{1}{\beta^2\, \epsilon(\omega)}\right) \cdot \Theta[\,\epsilon(\omega)\, \beta^2 - 1\,]$$

(13.48)

Check the units.

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Derivation

■ We start Maxwell's equations, for these sources

$\rho(\vec{x},t) = ze\,\delta^3\left(\vec{x} - \vec{v}\,t\right)$

$\vec{J}(\vec{x},t) = \vec{v}\,\rho(\vec{x},t)$

But it will be necessary to write Maxwell's equations in *reciprocal space* ($\vec{k}$ and ω).

$D(\vec{x},t) \neq \epsilon\, E(\vec{x},t)$; /important/ ; but rather $D(\vec{k},\omega) = \epsilon(\omega)\ \vec{E}(\vec{k},\omega)$.

■ Fourier transform conventions,

$F(\vec{x},t) = \int \frac{d^3 k}{(2\pi)^{3/2}} \int \frac{d\omega}{(2\pi)^{1/2}}\ \tilde{F}(\vec{k},\omega)\ \exp\{ i(\vec{k}\cdot\vec{x} - \omega t)\}$

To make the equations look simpler, drop the 'tilde' (~). This can be confusing, because F could mean $F(\vec{x},t)$ or $F(\vec{k},\omega)$. We need to keep in mind, are we talking about spacetime ($\vec{x}$ and time) or reciprocal space and frequency ($\vec{k}$ and ω).

<u>Example</u>

$\nabla\cdot\vec{D} = 4\pi\rho$ means $\nabla\cdot\vec{D}(\vec{x},t) = 4\pi\,\rho(\vec{x},t)$

space and time

$i\,\vec{k}\cdot\vec{D} = 4\pi\rho$ means $i\,\vec{k}\cdot\vec{D}(\vec{k},\omega) = 4\pi\,\rho(\vec{k},\omega)$

reciprocal space and frequency

In other words the second equation relates $\tilde{\vec{D}}$ and $\tilde{\rho}$ but dropping the tilde.

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■ Calculate $\rho(\vec{k},\omega)$

$$= \frac{1}{(2\pi)^2} \int d^3x \int dt \;\; ze\, \delta^3\left(\vec{x} - \vec{v}\,t\right) \; \exp\{-i\,(\vec{k}\cdot\vec{x} - \omega\, t)\}$$

$$= \frac{ze}{(2\pi)^2} \int dt\, \exp\{-i\,(\vec{k}\cdot\vec{v}\,t - \omega\, t)\}$$

$$= \frac{ze}{2\pi}\, \delta(\omega - \vec{k}\cdot\vec{v})$$

■ $\vec{J}(\vec{k},\omega) = \vec{v}\,\rho(\vec{k},\omega)$ because $\vec{v}$ is constant.

Maxwell's equations, in spacetime

$\nabla\cdot\vec{B} = 0$ and $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{c\,\partial t}$

homogeneous equations

$\nabla\cdot\vec{D} = 4\pi\rho$ and $\nabla\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{\partial\vec{D}}{c\,\partial t}$

inhomogeneous equations

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Maxwell's equations, in reciprocal space and frequency

$$i\,\vec{k}\cdot\vec{B} = 0 \quad \text{and} \quad i\,\vec{k}\times\vec{E} = \frac{i\omega}{c}\,\vec{B}$$

homogeneous equations

$$i\,\vec{k}\cdot\vec{D} = 4\pi\rho \quad \text{and} \quad i\,\vec{k}\times\vec{H} = \frac{4\pi}{c}\,\vec{J} - \frac{i\omega}{c}\,\vec{D}$$

inhomogeneous equations

where $\vec{D} = \epsilon(\omega)\,\vec{E}$ and $\vec{H} = \vec{B}/\mu(\omega)$.

For a dielectric, we'll just set $\mu(\omega) = 1$; i.e., $\vec{H} = \vec{B}$.

The Potentials

$\nabla\cdot\vec{B} = 0 \Longrightarrow$ write $\vec{B} = \nabla\times\vec{A}$;

$\nabla\times\vec{E} = -\,\partial\vec{B}/c\partial t = -\,\nabla\times(\partial\vec{A}/c\partial t)$

$$\Longrightarrow\ \vec{E} + \frac{\partial\vec{A}}{c\,\partial t} = -\,\nabla\Phi$$

In reciprocal space and frequency,

$$\vec{B} = i\,\vec{k}\times\vec{A}$$

$$\vec{E} = -\,i\,\vec{k}\,\Phi + \frac{i\omega}{c}\,\vec{A}$$

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Jackson's derivation of the Frank-Tamm equations

Jackson starts here — Eq. (13.22)

$$\left[k^2 - \frac{\omega^2}{c^2}\,\epsilon(\omega)\right]\Phi(\mathbf{k},\,\omega) = \frac{4\pi}{\epsilon(\omega)}\,\rho(\mathbf{k},\,\omega)$$

$$\left[k^2 - \frac{\omega^2}{c^2}\,\epsilon(\omega)\right]\mathbf{A}(\mathbf{k},\,\omega) = \frac{4\pi}{c}\,\mathbf{J}(\mathbf{k},\,\omega)$$

These are equations in reciprocal space, and are not obvious. So we'll go back a step and derive Eq. (13.22)

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To derive (13.22) we need the inhomogeneous equations. First,

$i\,\vec{k}\cdot\vec{D} = 4\pi\,\rho$ and $\vec{D} = \epsilon(\omega)\,\vec{E}$

$i\,\vec{k}\cdot\epsilon\,[\,-i\,\vec{k}\,\Phi + \frac{i\omega}{c}\,\vec{A}\,] = 4\pi\,\rho$

$\epsilon\,k^2\,\Phi = 4\pi\,\rho + \epsilon\,\frac{\omega}{c}\,\vec{k}\cdot\vec{A}$

$\epsilon\,[\,k^2 - \frac{\omega^2\,\epsilon}{c^2}\,]\,\Phi = 4\pi\,\rho + \epsilon\,\frac{\omega}{c}\,\{\,\vec{k}\cdot\vec{A} - \epsilon\,\frac{\omega}{c}\Phi\,\}$

Second,

$$i\,\vec{k}\times\vec{B} = \frac{4\pi}{c}\vec{J} - \frac{i\omega}{c}\vec{D}$$

$$i\,\vec{k}\times(i\,\vec{k}\times\vec{A}) = \frac{4\pi}{c}\vec{J} - \frac{i\omega}{c}\epsilon\left(-i\,\vec{k}\,\Phi + \frac{i\omega}{c}\vec{A}\right)$$

$$-\vec{k}\,(\vec{k}\cdot\vec{A}) + k^2\vec{A} = \frac{4\pi}{c}\vec{J} - \epsilon\frac{\omega}{c}\vec{k}\,\Phi + \frac{\omega^2}{c^2}\epsilon\vec{A}$$

$$\left[k^2 - \frac{\omega^2\epsilon}{c^2}\right]\vec{A} = \frac{4\pi}{c}\vec{J} + \vec{k}(\vec{k}\cdot\vec{A}) - \epsilon\frac{\omega}{c}\vec{k}\,\Phi$$

$$\left[k^2 - \frac{\omega^2\epsilon}{c^2}\right]\vec{A} = \frac{4\pi}{c}\vec{J} + \vec{k}\left\{\vec{k}\cdot\vec{A} - \epsilon\frac{\omega}{c}\Phi\right\}$$

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So, make this <u>gauge choice</u>

$$\vec{k}\cdot\vec{A} - \epsilon\frac{\omega}{c}\Phi = 0$$

Exercise: write the gauge choice in spacetime

$\Longrightarrow$ we have these wave equations

$$\left[k^2 - \frac{\omega^2\epsilon}{c^2}\right]\Phi = \frac{4\pi}{\epsilon}\rho$$

$$\left[k^2 - \frac{\omega^2\epsilon}{c^2}\right]\vec{A} = \frac{4\pi}{c}\vec{J}$$

(13.22)

The potentials

$$\Phi = \frac{4\pi}{\epsilon} \frac{\rho}{\left[k^2 - \frac{\omega^2 \epsilon}{c^2}\right]} = \frac{2\,ze}{\epsilon} \frac{\delta(\omega - \mathrm{k} \bullet \mathrm{v})}{\left[k^2 - \frac{\omega^2 \epsilon}{c^2}\right]}$$

$$\vec{A} = \frac{4\pi}{c} \frac{\vec{J}}{\left[k^2 - \frac{\omega^2 \epsilon}{c^2}\right]} = \frac{4\pi}{c} \frac{\vec{v}\,\rho}{\left[k^2 - \frac{\omega^2 \epsilon}{c^2}\right]} = \frac{\vec{v}}{c}\,\epsilon\,\Phi$$

The fields

$$\vec{E} = -i\,\vec{k}\,\Phi + \frac{i\omega}{c}\,\vec{A}$$

$$\vec{E}(\vec{k},\omega) = i\left[-\vec{k} + \frac{\omega\,\epsilon(\omega)}{c}\,\frac{\vec{v}}{c}\right]\Phi(\vec{k},\omega)$$

$$\vec{B}(\vec{k},\omega) = i\,\epsilon(\omega)\,\vec{k} \times \frac{\vec{v}}{c}\,\Phi(\vec{k},\omega)$$

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The radiated energy

Draw a picture.

Let $\vec{v} = \mathrm{v}\,\hat{e}_x$.

Consider points on the cylinder of radius b around the x axis. Say $\vec{x} = \mathrm{b}\,\hat{e}_y$ is the observation point. By cylindrical symmetry, the power is independent of azimuthal angle ϕ .

There the electric field, still in frequency space, is

$\vec{E}(\mathrm{b}\hat{e}_y, \omega) = \int \frac{d^3 k}{(2\pi)^{3/2}}\,\vec{E}(\vec{k},\omega)\,e^{i\,\vec{k} \bullet \vec{x}}$ where

$\vec{x} = \mathrm{b}\,\hat{e}_y$

Evaluate E_x

$$= \frac{2\,ize}{\epsilon\,(2\,\pi)^{3/2}} \int dk_1\, dk_2\, dk_3\, e^{i\,k_2\,b}$$

$$\cdot\left[\, \frac{\omega\,\epsilon\,v}{c^2} - k_1 \,\right] \frac{\delta(\omega - v k_1)}{k^2 - \epsilon\,\omega^2/c^2}$$

$$= \frac{2\,ize}{\epsilon\,(2\,\pi)^{3/2}} \int dk_2\, dk_3\, e^{i\,k_2\,b}$$

$$\cdot\left[\, \frac{\omega\,\epsilon\,v}{c^2} - \frac{\omega}{v} \,\right] \frac{1}{v}\; \frac{1}{k^2 - \epsilon\,\omega^2/c^2}$$

•

denominator = $k_2^2 + k_3^2 + \Lambda^2$

where $\Lambda^2 = \frac{\omega^2}{v^2} - \epsilon(\omega)\,\frac{\omega^2}{c^2} = \frac{\omega^2}{v^2}\,[\,1 - \epsilon(\omega)\,\beta^2\,]$

$$\Lambda^2 = \frac{\omega^2}{v^2}\,[\,1 - \epsilon(\omega)\,\beta^2\,]$$

•

The k_3 integral = $\int \frac{dk_3}{k_3^2 + k_2^2 + \Lambda^2} = \frac{\pi}{(k_2^2 + \Lambda^2)^{1/2}}$

```
In[3]:= (**)
Assuming[M > 0,
 Integrate[1 / (x^2 + M^2),
  {x, -Infinity, Infinity}]]
```

Out[3]= $\frac{\pi}{M}$

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The k2 integral

$$E_x = \frac{2\, i z e}{\epsilon\,(2\,\pi)^{3/2}}\,\pi\,\left[\,\frac{\omega\,\epsilon\,v}{c^2} - \frac{\omega}{v}\,\right]\frac{1}{v}\;\square$$

$$\square \int dk_2\; e^{i\,k_2\,b}\,\frac{1}{(k_2^2+\Lambda^2)^{1/2}}$$

In[8]:=
```
(**)
Assuming[μ > 0,
 Integrate[
  Cos[ξ] / (ξ^2 + μ^2)^(1/2),
  {ξ, -∞, ∞}]]
```
Out[8]=
```
2 BesselK[0, μ]
```

modified Bessel function

$$E_x = \frac{2\,i\,z e}{\epsilon\,(2\,\pi)^{3/2}}\,\pi\,\left[\,\frac{\omega\,\epsilon\,v}{c^2} - \frac{\omega}{v}\,\right]\frac{1}{v}\;2\;K_0(b\Lambda)$$

$$E_x = -\,\frac{i\,z e\,\omega}{\epsilon\,v^2}\left(\frac{2}{\pi}\right)^{1/2}\left[\,1 - \frac{\epsilon\,v^2}{c^2}\,\right]K_0(b\Lambda)$$

$$E_x(b e_y, \omega) = -\,\frac{i\,z e\,\omega}{v^2}\left(\frac{2}{\pi}\right)^{1/2}\left[\,\frac{1}{\epsilon(\omega)} - \beta^2\,\right]K_0(\,b\Lambda)$$

Other components (given by Jackson)

$$E_y(b e_y, \omega) = \frac{z e}{v}\left(\frac{2}{\pi}\right)^{1/2}\frac{\Lambda}{\epsilon(\omega)}\;K_1(\,b\Lambda)$$

$$B_z(b\,e_y\,,\omega) = \epsilon(\omega)\,\beta\,E_y(b\,e_y\,,\omega)$$

Draw a picture.

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Radiated energy

Calculate the energy flux per unit length radiated out of the cylinder of radius b.

Take the limit b ≫ a, where a is an atomic dimension , e.g., a ~ 10×10^{-8} cm.

So b Λ ≫ 1.

In[10]:=
```
(**)
Series[BesselK[0, x], {x, ∞, 2}]
```

Out[10]=

$$e^{-x+O\left[\frac{1}{x}\right]^3}\left(\sqrt{\frac{\pi}{2}}\sqrt{\frac{1}{x}}-\frac{1}{8}\sqrt{\frac{\pi}{2}}\left(\frac{1}{x}\right)^{3/2}+O\left[\frac{1}{x}\right]^{5/2}\right)$$

$$E_x(b,\omega) \sim i\,\frac{ze\omega}{c^2}\left[1-\frac{1}{\beta^2\,\epsilon(\omega)}\right]\frac{e^{-b\Lambda}}{\sqrt{b\Lambda}}$$

$$E_y(b,\omega) \sim \frac{ze}{v\epsilon(\omega)}\sqrt{\frac{\Lambda}{b}}\,e^{-b\Lambda}$$

$$B_z(b,\omega) \sim \beta\,\epsilon(\omega)\,E_y(b,\omega)$$

Energy per unit time passing through the cylinder is

$$\frac{dE}{dt} = \frac{c}{4\pi}\int 2\pi a\ dx\ \hat{e}_\rho\cdot(\vec{E}\times\vec{B})$$

The integral over all x at one instant of time is the same as the integral at one point on the cylinder over all time, so replace dx by v dt; then

$$\frac{dE}{dx} = \frac{1}{v}\ \frac{ca}{2} \int v\, dt\, E_\psi(t)\, B_z(t)$$

$$= \frac{ca}{2} \text{Re} \int dt\, E_\psi(t)\, B_z(t)^*$$

$$= ca\, \text{Re} \int_0^\infty d\omega\, E_\psi(\omega)\, B_z(\omega)^*$$

Far from the path of the charge,

$$ca\, E_\psi\, B_z{}^* = \frac{z^2\, e^2}{c^2} \left(-i \sqrt{\frac{\Lambda^*}{\Lambda}}\right)$$

$$\square\, \omega \left[1 - \frac{1}{\beta^2\, \epsilon(\omega)} \right] e^{-b\, (\Lambda+\Lambda^*)}$$

We need to take the real part and integrate over frequencies. If Λ has a real part, then the exponential will be negligible, for b far from the path. That is atomic radiation, not Cherenkov radiation.

The Cherenkov radiation comes from the domain where Λ is purely imaginary.

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The domain of purely imaginary Λ = Cherenkov radiation

$$\Lambda^2 = \frac{\omega^2}{v^2}\,[\,1 - \epsilon(\omega)\,\beta^2\,]$$

Suppose $\epsilon(\omega)$ is real (i.e., there is no absorption) and $\beta^2\epsilon(\omega) > 1$.

That is, $v > c/\sqrt{\epsilon} = c/n$ = the speed of light.

Then Λ is purely imaginary.

This is the Cherenkov radiation.

- $-i\sqrt{\Lambda^*/\Lambda} = 1$
- $\exp\{-b*(\Lambda+\Lambda^*)\} = 1$

$$\left(\frac{dE}{dx}\right)_{C.rad.} = \frac{z^2 e^2}{c^2}\int_{C.rad.} d\omega\,\omega\left(1 - \frac{1}{\beta^2\,\epsilon(\omega)}\right)$$

The domain of integration is where $\epsilon(\omega) > 1/\beta^2$.

The integrand is $d^2E/(dx\,d\omega)$.

This is the Frank-Tamm formula.