Consider again a charged particle moving with constant velocity. Calculate the fields.

If the particle is in empty space then there are three ways to calculate the fields.

1- Lorentz transformation from the rest frame.

2- Use the Lienard - Wiechert potentials.

3- Solve Maxwell's equations in reciprocal space, and then calculate the Fourier transform.

If the particle is moving in a dielectric, then we must use method #3; Jackson, Sections 13.3 and 13.4.

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I was not completely satisfied with the calculations in Jackson, so I'm going to do this over again today.

Maxwell's equations in reciprocal space ($\vec{k}$ and ω).

$$F(\vec{x}, t) = \int \frac{d^3k\, d\omega}{(2\pi)^2}\; F(\vec{k},\omega)\, \exp\{\, i\,(\vec{k}\cdot\vec{x} - \omega t)\,\}$$

$$i\,\vec{k}\cdot\vec{B} = 0 \text{ and } i\,\vec{k}\times\vec{E} = \frac{i\omega}{c}\vec{B}$$

$$i\,\vec{k}\cdot\vec{D} = 4\pi\rho \text{ and } i\,\vec{k}\times\vec{H} = \frac{4\pi}{c}\vec{J} - \frac{i\omega}{c}\vec{D}$$

$$\vec{D}(\vec{k},\omega) = \epsilon(\omega)\,\vec{E}(\vec{k},\omega) \text{ and } \vec{H}(\vec{k},\omega) = \vec{B}(\vec{k},\omega)/\mu(\omega)$$

Potentials, $\vec{A}(\vec{k},\omega)$ and $\Phi(\vec{k},\omega)$;

$$\vec{B} = i\,\vec{k}\times\vec{A} \text{ and } \vec{E} = -i\,\vec{k}\,\Phi + \frac{i\omega}{c}\vec{A}$$

Impose a gauge choice on the potentials,

$i\,\vec{k}\cdot A + \frac{i\omega}{c} n^2\,\Phi = 0$ where $n^2 = \epsilon(\omega)\,\mu(\omega)$;

in the case of free space ($\epsilon=\mu=1$) this is Lorenz gauge condition.

Then rewrite the inhomogeneous equations in terms of the potentials;

$\Longrightarrow$

$$[\,k^2 - \frac{\omega^2}{c^2}\, n^2]\, \Phi = \frac{4\,\pi}{\epsilon}\, \rho$$

$$[\,k^2 - \frac{\omega^2}{c^2}\, n^2]\, \vec{A} = \frac{4\,\pi}{c}\, \vec{J}$$

Sources, potentials and fields for a particle moving with constant velocity

$$\rho(\,\vec{x},\mathrm{t}) = \mathrm{e}\;\delta^3(\,\vec{x} - \vec{v}\;\mathrm{t})$$

$$\Rightarrow \rho(\,\vec{k},\omega) = \frac{e}{2\,\pi}\;\delta(\,\vec{v}\bullet\vec{k} - \omega\,)$$

$$\Phi = \frac{4\,\pi}{\epsilon}\;\frac{\rho}{\left[k^2 - \frac{\omega^2}{c^2}\,n^2\right]} = \frac{2\,e}{\epsilon}\;\frac{\delta\left(\vec{v}.\vec{k}-\omega\right)}{\left[k^2 - \frac{\omega^2}{c^2}\,n^2\right]}$$

$$\vec{J}(\,\vec{x},\mathrm{t}) = \vec{v}\;\rho(\,\vec{x},\mathrm{t})$$

$$\Rightarrow \vec{J}(\,\vec{k},\omega) = \vec{v}\;\rho(\,\vec{k},\omega)$$

$$\vec{A} = \frac{4\,\pi}{c}\;\frac{\vec{J}}{\left[k^2 - \frac{\omega^2}{c^2}\,n^2\right]} = \frac{\epsilon}{c}\;\vec{v}\;\Phi$$

$$\vec{E}(\,\vec{k},\,\omega) = -\mathrm{i}\;\vec{k}\;\Phi + \frac{i\,\omega}{c}\;\vec{A} = (-\mathrm{i}\;\vec{k} + \frac{i\,\omega\,\epsilon}{c^2}\;\vec{v}\,)\;\Phi$$

and

$$\vec{B}(\,\vec{k},\,\omega) = \mathrm{i}\;\vec{k}\times\vec{A} = \frac{i\,\epsilon}{c}\;\vec{k}\times\vec{v}\;\;\Phi$$

Calculation of the electric field

$$\vec{E}(\,\vec{k},\,\omega) = -\mathrm{i}\;\vec{k}\;\Phi + \frac{i\,\omega}{c}\;\vec{A} = (-\mathrm{i}\;\vec{k} + \frac{i\,\omega\,\epsilon}{c^2}\;\vec{v}\,)\;\Phi$$

$$\vec{E}\;(\,\vec{x},\,\mathrm{t}) = \int \frac{d^3\,k\,\mathrm{d}\omega}{(2\,\pi)^2}\;\vec{E}(\,\vec{k},\omega)\;\exp\{\,\mathrm{i}\,(\,\vec{k}\bullet\,\vec{x} - \omega\,\mathrm{t}\,)\,\}$$

Velocity is constant, so let $\vec{v} = \mathrm{v}\,\hat{e}_x$.

W.L.O.G., Let the observation point be $\vec{x} = \{\, \mathrm{x, y, 0} \,\}$; there is no loss of generality because by cylindrical symmetry, the electric field at

$\{\, \mathrm{x},\ \mathrm{b}\cos\psi,\ \mathrm{b}\sin\psi \,\}$ does not depend on ψ; so we only need to calculate the point $\vec{x}$ at $\psi = 0$.

$$\vec{E}\,(\vec{x}\,;\omega) = \int \frac{d^3 k}{(2\,\pi)^{3/2}}\,\vec{E}(\,\vec{k},\omega)\,\exp\{\,\mathrm{i}\,\vec{k}\bullet\vec{x}\,]$$

$$\vec{E}\,(\vec{x}\,;\omega) = \int \frac{d^3 k}{(2\,\pi)^{3/2}}\,\exp\{\,\mathrm{i}\,\vec{k}\bullet\vec{x}\,]\ \square$$

$$\square\ \left(-i\,\vec{k} + \frac{i\,\omega\epsilon}{c^2}\,\vec{v}\,\right)\,\frac{2\,e}{\epsilon}\,\frac{\delta(v.k-\omega)}{\left[k^2 - \frac{\omega^2}{c^2}\,n^2\right]}$$

$$\vec{E}\,(x,\ y,\ o\,;\omega) = \frac{2\,e}{\epsilon}\int \frac{d^3 k}{(2\,\pi)^{3/2}}\,\exp\{\,\mathrm{i}\,k_1 \mathrm{x}\,]\,\exp[\,\mathrm{i}\,k_2\,\mathrm{y}]\ \square$$

$$\square\ \left(-i\,\vec{k} + \frac{i\,\omega\epsilon}{c^2}\,\mathrm{v}\,\hat{e}_x\,\right)\,\frac{\delta(v\,k_1-\omega)}{\left[k^2 - \frac{\omega^2}{c^2}\,n^2\right]}$$

$$\vec{E}\,(x,\ y,\ o\,;\omega) = \frac{2\,e}{\epsilon\,v}\int \frac{\mathrm{dk}_2\,\mathrm{dk}_3}{(2\,\pi)^{3/2}}\,\exp\{\,\mathrm{i}\,\omega\,\mathrm{x/v}\,]\,\exp[\,\mathrm{i}\,k_2\,\mathrm{y}]\ \square$$

$$\square\ [\,\hat{e}_x\left(-i\,\frac{\omega}{v} + \frac{i\,\omega\epsilon}{c^2}\,\mathrm{v}\,\right) - \mathrm{i\,k2}\,\hat{\mathrm{e}}\mathrm{y} - \mathrm{i\,k3}\,\hat{\mathrm{e}}\mathrm{z}\,]\ \frac{1}{k_2^2 + k_3^2 + \Lambda^2}$$

$$\Lambda^2 = (\omega/v)^2 - (\omega/c)^2\,n^2$$

$$\Lambda = \omega\,\lambda$$

$$\lambda = (\,1/v^2 - n^2/c^2\,)^{1/2}$$

Do the k_3 integral ...

$$\vec{E}(x, y, o; \omega) = \frac{2e}{\epsilon v} \int \frac{dk_2}{(2\pi)^{3/2}} \exp\{i\omega x/v\} \exp[i k_2 y] \,\cdot$$

$$\cdot \left[\hat{e}_x \left(-i \frac{\omega}{v} + \frac{i\omega\epsilon}{c^2} v \right) - i k_2 \hat{e}_y \right] \frac{\pi}{\sqrt{k_2^2 + \Lambda^2}}$$

Do the k_2 integral ...

$$\vec{E}(x, y, o; \omega) = \frac{2e}{\epsilon v} \frac{\pi}{(2\pi)^{3/2}} \exp\{i\omega x/v\} \,\cdot$$

$$\cdot \left[\hat{e}_x \left(-i \frac{\omega}{v} + \frac{i\omega\epsilon}{c^2} v \right) 2 K_0(\Lambda y) - \hat{e}_y 2 \Lambda K_0'(\Lambda y) \right]$$

$$\vec{E}(x, y, o; \omega) = \frac{-ie}{\epsilon v} \sqrt{\frac{2}{\pi}} \exp\{i\omega x/v\}$$

$$\cdot \left[\hat{e}_x \left(\frac{\omega}{v} - \frac{\omega\epsilon}{c^2} v \right) K_0(\Lambda y) - i\, \hat{e}_y \Lambda K_1(\Lambda y) \right]$$

Fourier transform from frequency to time

$$\vec{E}(x, y, o; t) = \int \frac{d\omega}{\sqrt{2\pi}} \exp[-i\omega t] \vec{E}(x, y, o; \omega)$$

Real part is implied.

$$= \frac{-ie}{v} \sqrt{\frac{2}{\pi}} \int \frac{d\omega}{\epsilon} \exp[-i\omega(t - x/v)]$$

$$\cdot \left\{ \hat{e}_x \left(\frac{\omega}{v} - \frac{\omega\epsilon}{c^2} v \right) K_0(\Lambda y) - i\, \hat{e}_y \Lambda K_1(\Lambda y) \right]$$

$$\Lambda^2 = (\omega/v)^2 - (\omega/c)^2 n^2 ; \quad \Lambda = \frac{\omega}{v} \sqrt{1 - v^2 n^2 / c^2}$$

This result , together with $\vec{B}$(x,y,o;t) , leads to the Frank-Tamm formula for the frequency distribution of Cherenkov radiation in a dielectric; see the last lecture or Jackson Section 13.4.

Now consider a particle moving in ***free space***,
with $\vec{v} = v\,\hat{e}_x$, at $\vec{x} = \{x, y, 0\}$.

Now let $\epsilon = \mu = n = 1. \Longrightarrow$

$$\Lambda = \frac{\omega}{v}\sqrt{1 - v^2/c^2} = \frac{\omega}{\gamma v} = \frac{\omega c}{\beta\gamma}$$

$$\hat{e}_x\left(\frac{\omega}{v} - \frac{\omega}{c^2}v\right) = \hat{e}_x\frac{\omega}{v}(1-\beta^2) = \hat{e}_x\frac{\omega}{v\,\gamma^2}$$

$$\vec{E}(x, y, o; t) = \frac{-ie}{v}\sqrt{\frac{2}{\pi}}\int d\omega \exp[-i\,\omega\,(t - x/v)] \cdot$$

$$\cdot \left\{ \hat{e}_x \frac{\omega}{v\,\gamma^2} K_0\!\left(\frac{cy}{\beta\gamma}\omega\right) - i\,\hat{e}_y \frac{c}{\beta\gamma}\omega K_1\!\left(\frac{cy}{\beta\gamma}y\right) \right]$$

Let $\alpha = t - x/v$.

Take the real part.

$\exp[-i\omega\alpha] \longrightarrow -i\sin(\omega\alpha)$; or, $\exp[-i\omega\alpha] \longrightarrow \cos(\omega\alpha)$

Some Mathematica Calculations

In[42]:=
```
Assuming[Λ ∈ Reals && y ∈ Reals && Λ > 0,
  Integrate[Cos[k2 * y] / Sqrt[k2^2 + Λ^2],
   {k2, 0, Infinity}] * 2]
```
Out[42]=
```
2 BesselK[0, Λ Abs[y]]
```

In[43]:=
```
D[BesselK[0, x], x]
```
Out[43]=
```
-BesselK[1, x]
```

```
Assuming[α ∈ Reals && μ ∈ Reals && μ > 0,
  Integrate[ω * Sin[ω * α] BesselK[0, ω * μ],
   {ω, 0, Infinity}]] * 2
```
Out[44]=

$$\frac{\pi\,\alpha}{\left(\alpha^2 + \mu^2\right)^{3/2}}$$

In[45]:=
```
Assuming[α ∈ Reals && μ ∈ Reals && μ > 0,
  Integrate[ω * Cos[ω * α] BesselK[1, ω * μ],
   {ω, 0, Infinity}]] * 2
```

Out[45]= $\frac{\pi \mu}{\left(\alpha^2 + \mu^2\right)^{3/2}}$

$$\vec{E}\,(x, y, o\,;\mathrm{t}\,) = \frac{-i\,e}{v} \sqrt{\frac{2}{\pi}} \int d\omega \exp[-\mathrm{i}\,\omega\,(\mathrm{t}-\mathrm{x/v})\,] \,\square$$

$$\square\,\{\ \hat{\mathrm{e}}_x \frac{\omega}{v\,\gamma^2}\ K_0\!\left(\frac{\mathrm{cy}}{\beta\gamma}\,\omega\right) - \mathrm{i}\ \hat{\mathrm{e}}_y \frac{c}{\beta\gamma}\,\omega\,K_1\!\left(\frac{\mathrm{cy}}{\beta\gamma}\,\omega\right)\,]$$

$$\vec{E}\,(x, y, o\,;\mathrm{t}\,) = \frac{-i\,e}{v} \sqrt{\frac{2}{\pi}}$$

$$\square\,\{\ -\mathrm{i}\ \hat{\mathrm{e}}_x \frac{1}{v\,\gamma^2}\ \frac{\pi\,\alpha}{(\,\alpha^2+\mu^{2/})^{3/2}}\ -\ \mathrm{i}\ \hat{\mathrm{e}}_y \frac{c}{\beta\gamma}\ \frac{\pi\,\mu}{(\,\alpha^2+\mu^{2/})^{3/2}}\ \}$$

where $\alpha = \mathrm{t} - \mathrm{x/v}$ and $\mu = \mathrm{cy}/(\beta\gamma)$.

Exercise: Find and correct the errors in the previous equation.

Another page of algebra ...

$$\vec{E}\,(x, y, o\,;\mathrm{t}\,) = \mathrm{e}\,\gamma\ \frac{\hat{\mathrm{e}}_x(x-\mathrm{vt}) + \hat{\mathrm{e}}_y\,y}{[\ \gamma^2(x-\mathrm{vt})^2 + y^2\ ]^{3/2}}$$

This agrees with the Lorentz transformation from the rest frame of the particle.

Note that the ***direction*** of the electric field at time t points directly away from the position of the particle at that time, {vt,0,0}. It does not point away from the position of the particle at the retarded time! The electric field 9and also the magnetic field) travels along with the particle.

What about the ***magnitude***?

For a given y, the strength of the field is strongest at x = vt;

$\vec{E}$ (vt , b, 0 ; t) = $\gamma \; \frac{e}{b^2} \, \hat{e}_y$.

For a given x – vt, the strength of the field is weakest at y = 0;

$\vec{E}$ (vt + b , 0, 0 ; t) = $\frac{1}{\gamma} \frac{e}{b^2} \, \hat{e}_x$.