Bremsstrahlung

Jackson, Chapter 15
Wilcox and Thron, Section 11.9

Jackson, Section 15.1:
Radiation Emitted during Collisions

In[1850]:= eq["15.1.png"]

Out[1850]=
$$\frac{dI(\omega)}{d\Omega} = \frac{e^2}{4\pi^2 c}\left|\int \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})e^{i\omega\left(t-\frac{\mathbf{n}\cdot\mathbf{r}(t)}{c}\right)}\,dt\right|^2$$

15.1

In[1851]:= eq["15.2.png"]

Out[1851]=
$$\frac{dI(\omega)}{d\Omega} = \frac{e^2}{4\pi^2 c}\left|\int \mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})e^{i\omega t}\,dt\right|^2$$

15.2

For low frequencies ...

$$\int dt\, \dot{\boldsymbol{\beta}} \approx \Delta\vec{\beta} = \vec{\beta}_2 - \vec{\beta}_1$$

In[1852]:= eq["15.5.png"]

Out[1852]=
$$\frac{dI(\omega)}{d\Omega} \simeq \frac{e^2}{4\pi^2 c}|\Delta\boldsymbol{\beta}|^2 \sin^2\Theta, \qquad \omega\tau \ll 1$$

15.5

$$\hat{n}\bullet\Delta\vec{\beta} = |\Delta\vec{\beta}|\cos\Theta$$

In[1853]:= eq["15.6.png"]

Out[1853]=
$$I(\omega) \simeq \frac{2}{3\pi}\frac{e^2}{c}|\Delta\boldsymbol{\beta}|^2, \qquad \omega\tau \ll 1$$

15.6

For high frequencies ...
I(ω) is negligible.

In[1861]:=
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eq["fig15.1.png"]
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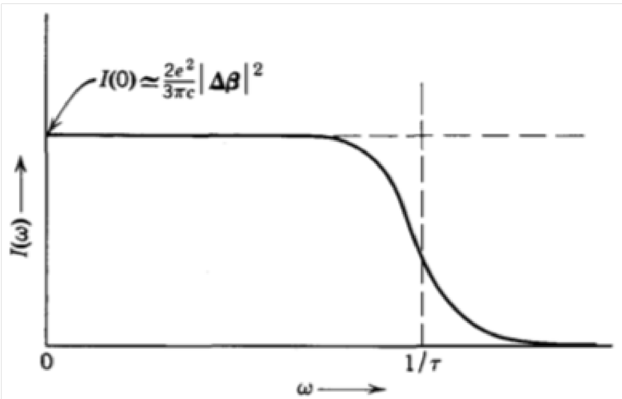
Out[1861]=



Rough approximation,

In[1862]:=
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eq["15.7.png"]
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Out[1862]=

$$I(\omega) = \begin{cases} \frac{2e^2}{3\pi c}|\Delta\boldsymbol{\beta}|^2, & \omega\tau < 1 \\ 0, & \omega\tau > 1 \end{cases}$$

15.7

Jackson, Section 15.2
Bremsstrahlung in Nonrelativistic Coulomb Collisions

Nonrelativistic Coulomb collisions are very familiar: it's *Rutherford scattering*. Let ze = the charge of the light projectile and Ze = the charge of the heavy target. The projectile moves along a hyperbola.

In[2047]:=
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eq["13.prob1.png"]
eq["13.1.png"]
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Out[2047]=

$$b = \frac{ze^2}{pv} \cot\frac{\theta}{2}$$

Out[2048]=

$$\frac{d\sigma}{d\Omega} = \left(\frac{ze^2}{2pv}\right)^2 \operatorname{cosec}^4 \frac{\theta}{2}$$

In the case of a large impact parameter ...

- the trajectory is approximately a straight line (θ is small);
- the change of momentum $\overrightarrow{\Delta p}$ is small and it is *transverse* ;

- $|\overrightarrow{\Delta p}| = \dfrac{2\, zZ\, e^2}{bv}$.

Proof:

$\vec{p}_i = Mv\, \hat{e}_x$

$\vec{p}_f = Mv\,(\hat{e}_x \cos\theta - \hat{e}_y \sin\theta) \approx Mv\,(\hat{e}_x - \hat{e}_y\, \theta)$

$\overrightarrow{\Delta p} \approx -\hat{e}_y\, Mv\, \theta$

and

$b \approx \dfrac{zZe^2}{Mv^2}\, \dfrac{2}{\theta} \approx \dfrac{2\, zZ\, e^2}{Mv^2}\, \dfrac{Mv}{|\Delta p|}$

$|\Delta p| = \dfrac{2\, zZ\, e^2}{bv}$

Then in the nonrelativistic approximation,

In[1863]:= eq["15.10.png"]

Out[1863]=

$$\Delta v = \frac{2zZe^2}{Mvb}$$

15.10

Plug that into Eq. 15.7 $\Longrightarrow$

In[1864]:= eq["15.11.png"]

Out[1864]=

$$I(\omega, b) \simeq \begin{cases} \frac{8}{3\pi}\left(\frac{z^2e^2}{Mc^2}\right)^2 \frac{Z^2e^2}{c}\left(\frac{c}{v}\right)^2 \frac{1}{b^2}, & \omega < \frac{v}{b} \\ 0, & \omega > \frac{v}{b} \end{cases}$$

15.11

In[2031]:=

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(* check *)
Δv = 2 z * Z * e^2 / (M * v * b);
Δβ = Δv / c;
intensity = 2 z^2 * e^2 / (3 π * c) * (Δβ)^2 // Simplify
compare = 8 / (3 π) * (z^2 * e^2 / (M * c^2))^2 *
    (Z^2 * e^2) / c * (c / v)^2 / b^2 // Simplify
```

Out[2033]=

$$\frac{8\, e^6\, z^4\, Z^2}{3\, b^2\, c^3\, M^2\, \pi\, v^2}$$

Out[2034]=

$$\frac{8\, e^6\, z^4\, Z^2}{3\, b^2\, c^3\, M^2\, \pi\, v^2}$$

Define the *radiation cross section* $\chi(\omega)$ by

$$\chi(\omega) = \int_{b_{\min}}^{b_{\max}} I(\omega,b)\, 2\pi\, b\, db$$

check the units:
(energy/frequency)·area

Note that $I(\omega,b) \sim 1/b^2$ as $b \longrightarrow 0$; so the integral is logarithmically divergent as $b \to 0$. To continue we *need* a lower limit on b, $b_{\min}$.
There is also an upper limit, $b_{\max}$.
Now, what are these limits, $b_{\min}$ and $b_{\max}$?

- $b_{\min}^{(c)} \simeq \dfrac{zZe^2}{Mv^2}$

Explain:
$b = \frac{zZe^2}{pv} \cot \frac{\theta}{2}$ *and we are only considering small* θ;
say $\theta < \pi/2$;
then $b > \frac{zZ\,e^2}{Mv^2}$ $\longleftarrow$ *min b*

- $b_{\max}^{(c)} \simeq \dfrac{v}{\omega}$

Explain:
Eq 15.11 $\Rightarrow \omega < v/b$ *for Brems. radiation;*
$\therefore b < v/\omega \longleftarrow$ *max b*

Now evaluate the integral (approximately)

In[1865]:=
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eq["15.15.png"]
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Out[1865]=

$$\chi_c(\omega) \simeq \frac{16}{3}\frac{Z^2e^2}{c}\left(\frac{z^2e^2}{Mc^2}\right)^2\left(\frac{c}{v}\right)^2 \ln\left(\frac{\lambda M v^3}{zZe^2\omega}\right)$$

15.15

Here λ is a numerical factor of order 1.

χ_c is a *classical estimate* for the cross section.

In Eq. 15.15 we have assumed that the logarithm is large compare to 1, which requires $b_{\max} \gg b_{\min}$.

Requiring $b_{\max} \gg b_{\min}$ implies an upper limit on frequency,

$$\omega_{\max}^{(c)} \sim \frac{Mv^3}{zZe^2}$$

Explain:

$$\frac{v}{\omega} \gg \frac{zZ\,e^2}{Mv^2}$$

$$\omega \ll \frac{Mv^3}{zZ\,e^2}$$

Quantum modifications

The incident particle (charge ze) is really a wave.

Or, more precisely, the incident state is defined by a probability wave. The deBroglie wavelength is

$$\lambda_q = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{Mv};$$

so, b ≫ λ implies b > b_{min} , where

$$b_{min}^{(q)} \simeq \frac{\hbar}{Mv}$$

Using that in the integral over b,

In[1866]:=
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eq["15.18.png"]
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Out[1866]=

$$\chi_q(\omega) \simeq \frac{16}{3}\frac{Z^2e^2}{c}\left(\frac{z^2e^2}{Mc^2}\right)^2\left(\frac{c}{v}\right)^2 \ln\left(\frac{\lambda M v^2}{\hbar\omega}\right)$$

15.18

The function $\chi_q(\omega)$ is a *semi-classical estimate* of the cross section.

Also, in the semi-classical theory, the upper limit *on the frequency* should be modified to (ω < v/b)

$$\omega_{max}^{(q)} \sim \frac{v}{b_{min}} \sim \frac{Mv^2}{\hbar} .$$

Also, conservation of energy requires $\omega < E/\hbar = Mv^2/2\hbar$.

Generally, $\omega_{max}^{(c)} \ll \omega_{max}^{(q)}$. Therefore ...

In[1867]:=
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eq["15.20.par.png"]
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Out[1867]=

This shows that the classical frequency spectrum is always confined to very low frequencies compared to the maximum allowed by conservation of energy. Thus the classical domain is of little interest. In what follows we will concentrate on the quantum-mechanical results.

Superior quantum modifications

We have,

$b_{\max} \simeq \frac{V}{\omega}$ and $b_{\min} \simeq \frac{\hbar}{\mathrm{Mv}}$.

Replace v by the average, $\langle v \rangle \equiv (v_i + v_f)/2$

$\langle v \rangle = \sqrt{\frac{E}{2M}} + \sqrt{\frac{E - \hbar\omega}{2M}}$ where $E = (1/2)\,\mathrm{Mv}^2$.

In[1868]:= eq["15.22.png"]

Out[1868]=

$$\chi_q(\omega) \simeq \frac{16}{3}\frac{Z^2e^2}{c}\left(\frac{z^2e^2}{Mc^2}\right)^2\left(\frac{c}{v}\right)^2 \ln\left[\frac{\lambda}{2}\frac{(\sqrt{E} + \sqrt{E - \hbar\omega})^2}{\hbar\omega}\right]$$

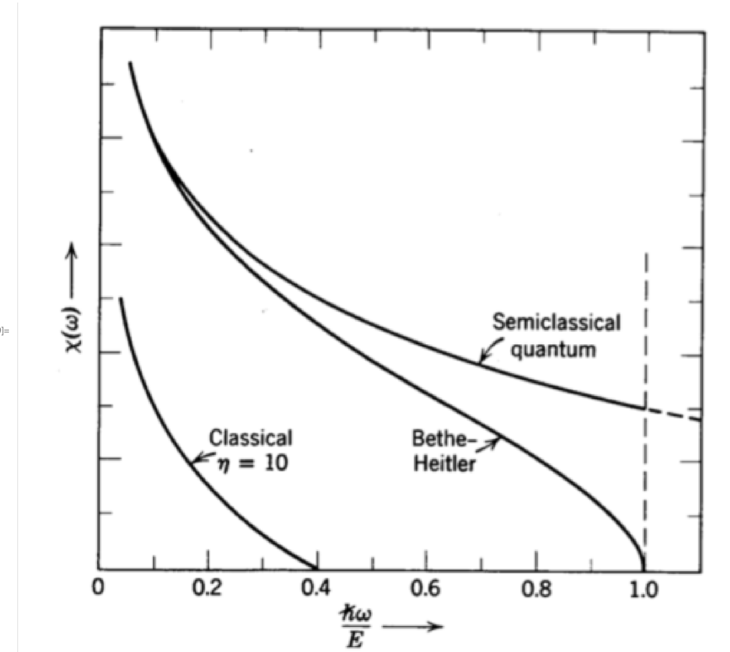
15.22

In fact, set $\lambda = 2$.

Then Eq. 15.22 is a fully quantum formula; derived using quantum electrodynamics by Bethe and Heitler in 1934.
(based on the Born approximation)

In[1869]:= eq["fig.15.3.png"]

Out[1869]=



Check the graphics

Let $\rho = \hbar\omega/E$ where $E = \frac{1}{2}M\, v^2 = \frac{1}{2}\, Mc^2\, \beta^2$.

Plot χ versus ρ.

■ -- Aside --

Classical case $\propto\ \ln\{\lambda M v^3/(zZ\, e^2\, \omega)\}$;

Jackson uses $\eta = ze^2/(\hbar v) = 10$;

$\Longrightarrow \ln[\ \lambda\ 2E\ v\ /(\ Z\ \eta\ \hbar v\ \omega)$

$= \ln[\ (2\lambda)\ \cdot(Z\eta)^{-1}\ \cdot(1/\rho)\ ]\ \longrightarrow \ln[\ 0.4/\rho\ ]$

■ -- --------- --

In[2027]:=
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χBH[ρ_] = Log[( Sqrt[1 / ρ] + Sqrt[ 1 / ρ - 1] )^2];
χSC[ρ_] = Log[4 / ρ];
χCL[ρ_] = Log[0.4 / ρ];
Plot[ {χBH[r] , χSC[r], χCL[r]}, {r, 0.05, 1},
  PlotRange → {{0, 1.1}, {0, 5}},
  PlotStyle → {
     {Thickness[0.01], Red},
     {Dashing[{0.03, 0.03}], Thickness[0.0075], Purple},
     {Dashing[{0.03, 0.02}], Thickness[0.0075], Blue}},
  Frame → True, FrameLabel → {"ħω/E", "χ(ω)"},
  BaseStyle → {24}, AspectRatio → 1, ImageSize → 640]
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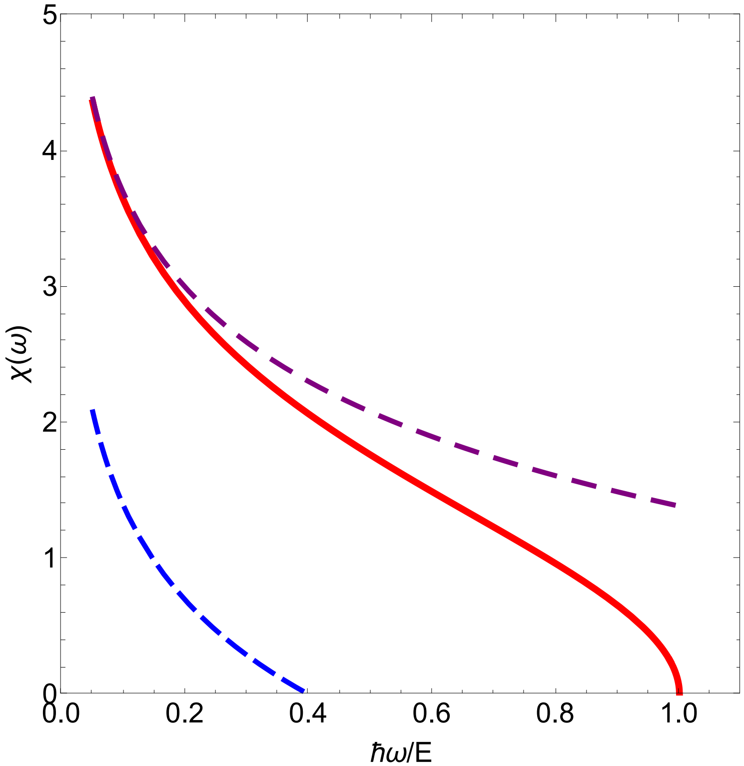
Out[2030]=



Other results in Jackson, Section 15.2

In[1870]:=
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eq["15.22.png"]
eq["15.24.png"]
eq["15.25.png"]
eq["15.26.png"]
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Out[1870]=

$$\chi_q(\omega) \simeq \frac{16}{3}\frac{Z^2e^2}{c}\left(\frac{z^2e^2}{Mc^2}\right)^2\left(\frac{c}{v}\right)^2 \ln\left[\frac{\lambda}{2}\frac{(\sqrt{E}+\sqrt{E-\hbar\omega})^2}{\hbar\omega}\right]$$

Out[1871]=

$$\sigma_{\text{brems}}(\hbar\omega) \simeq \frac{16}{3}\frac{Z^2e^2}{\hbar c}\left(\frac{z^2e^2}{Mc^2}\right)^2\left(\frac{c}{v}\right)^2 \ln\left(\frac{\quad}{\hbar\omega}\right)$$

Out[1872]=

$$\frac{dE_{\text{rad}}}{dx} = N\int_0^{\omega_{\max}} \chi(\omega)\, d\omega$$

Out[1873]=

$$\frac{dE_{\text{rad}}}{dx} = \frac{16}{3} NZ\left(\frac{Ze^2}{\hbar c}\right)\frac{z^4e^4}{Mc^2}\int_0^1 \ln\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)dx$$

```
Integrate[
  Log[(1 + Sqrt[1 - x]) / Sqrt[x]], {x, 0, 1}]
```

ut[1874]=

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1
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