PHY 842 SUMMARY

The purpose of a course like PHY 842 is to learn...

- first principles
- math methods
- examples

The course had many subjects:

CH9 Electromagnetic waves & optics

+ Frequency dependence of polarization; $\epsilon(\omega)$

CH11 Radiation by systems; $\vec{J}(x,t) \longrightarrow dP/d\Omega$

+ Radiation by charges; $\vec{r}(t) \longrightarrow \vec{E}$ and $\vec{B} \longrightarrow dP/d\Omega$

→ general results
→ Larmor's formula
→ synchrotron radiation
→ Cherenkov radiation
→ Bremsstrahlung

CH12 Scattering of light

+ Diffraction of light

FIRST PRINCIPLES FOR RADIATION CALCULATIONS

Given $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$,
what are the fields?
We could go back to the VERY FIRST first principles, i.e., Maxwell’s equations (in free space). But instead we’ll use general things that we know:

- potentials $\Phi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$;
- that guarantees the *homogeneous* Maxwell equations;
- then the *inhomogeneous* Maxwell equations imply that Φ and $\vec{A}$ obey the wave equation, with sources ρ and $\vec{J}$, in the Lorentz (or, Lorenz) gauge;
- the retarded Green’s function solves the wave equation.

⟹ The first principles for radiation calculations,

$$\Phi(\vec{x},t) = \int \frac{d^3x'}{\left|\vec{x}-\vec{x}'\right|} \rho\left(\vec{x}', t - \left|\vec{x}-\vec{x}'\right|/c\right)$$

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int \frac{d^3x'}{\left|\vec{x}-\vec{x}'\right|} \vec{J}\left(\vec{x}', t - \left|\vec{x}-\vec{x}'\right|/c\right)$$

gaussian units

retarded time = $t - |\vec{x} - \vec{x}'|/c$

Radiation by "systems"

for example, an antenna

One method is to consider *harmonic sources*

$\vec{J}(\vec{x},t) = \vec{J}(\vec{x})\, e^{-i\omega t}$

$\rho(\vec{x},t) = \rho(\vec{x})\, e^{-i\omega t}$

$\nabla \cdot \vec{J}(\vec{x}) = i\,\omega\,\rho(\vec{x})$

$\vec{J}(\vec{x})$ and $\rho(\vec{x})$ are complex; eventually, take the real part.

$\Longrightarrow$

$\vec{A}(\vec{x},t) = \vec{A}(\vec{x})\, e^{-i\omega t}$

$\Phi(\vec{x},t) = \Phi(\vec{x})\, e^{-i\omega t}$

ditto

$$\vec{A}(\vec{x}) = \frac{1}{c} \int \frac{d^3 x'}{\left|\vec{x} - \vec{x}\,'\right|} \vec{J}(\vec{x}\,')\, e^{i k \left|\vec{x} - \vec{x}\,'\right|}$$

where k = ω/c.

Potentials in the far zone ($r \longrightarrow \infty$)

$$|\vec{x} - \vec{x}'| = \sqrt{r^2 - 2\,\vec{x}\cdot\vec{x}' + (r')^2}$$

$$= r\,\{\,1 - 2\,\hat{n}\cdot\vec{x}' + (r'/r)^2\,\}^{1/2}$$

$= r - \hat{n}\cdot\vec{x}'$ + higher order terms

$\longrightarrow$ the multipole expansion

Dipole Radiation ; $|\vec{x} - \vec{x}'| \longrightarrow r$;

This means that the retarded time t_{ret} is set equal to $t - r/c$, which may be regarded as an approximation ($r \to \infty$ or source size $\to$ 0).

$$\vec{A}(\vec{x}) = \frac{e^{ikr}}{cr} \int d^3x'\, \vec{J}(\vec{x}')$$

Exercise

$$\int d^3x'\, \vec{J}(\vec{x}') = -\,i\,\omega\,\vec{p}$$

where $\vec{p} = \int d^3x'\, \vec{x}'\, \rho(\vec{x}')$

i.e., $\vec{p}(t') = \int d^3x'\, \vec{x}'\, \rho(\vec{x}')\, e^{-i\,\omega\,t'}$

Fields and radiated power in the dipole approximation

$$\vec{A}(\vec{x}) = \frac{e^{ikr}}{cr} (- i \, \omega \, \vec{p})$$

$$\vec{B} = \vec{B}(\vec{x}) \, e^{-i\omega t}$$

$$\vec{B}(\vec{x}) = \nabla \times \vec{A}(\vec{x})$$

$$= -ik \frac{e^{ikr}}{cr} \left(ik - \frac{1}{r} \right) (\hat{n} \times \vec{p})$$

$$\approx k^2 \frac{e^{ikr}}{cr} (\hat{n} \times \vec{p})$$

What about $\vec{E}$?

$\vec{E} \approx \vec{B} \times \hat{n}$ in the far zone,

because the fields approach a plane wave in any direction *in the far zone*.

Radiated power in the dipole approximation

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B},$$

but we need to be careful with complex functions!

What we can say is this ...

$$\left\langle \frac{dP}{d\Omega} \right\rangle = r^2 \frac{c}{8\pi} \text{Re} \, \hat{n} \cdot (\vec{E} \times \vec{B}^*)$$

Exercise.

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^4}{32\pi^2 c^2} | \vec{p} |^2 \sin^2\theta$$

$$P = \frac{\omega^4}{12\pi} | \vec{p} |^2$$

power in dipole radiation

Radiation by a particle

The trajectory is given, $\vec{r}(t)$.

$\Longrightarrow$ The charge and current densities are

$\rho(\vec{x},t) = e\, \delta^3[\, \vec{x} - \vec{r}(t)\,]$

$\vec{J}(\vec{x},t) = e\, \dot{\vec{r}}(t)\, \delta^3[\, \vec{x} - \vec{r}(t)\,]$

$\Longrightarrow$ the potentials are

$$\vec{A}(\vec{x},t) = \frac{e}{c} \int dt' \, \frac{\dot{\vec{r}}(t')}{R(t')}\, \delta\,[\, t' - t + R(t')/c\,]$$

where $R(t') = |\, \vec{x} - \vec{r}(t')\, |$

$\Longrightarrow$ the Lienard-Wiechert potentials

$\Phi(\vec{x}\,,t) = e\left[\frac{1}{\kappa R}\right]_{\rm ret}$ where $\kappa = 1 - \hat{n} \cdot \vec{v}/c$

$\vec{A}(\vec{x}\,,t) = \frac{e}{c}\left[\frac{v}{\kappa R}\right]_{\rm ret}$

retarded time

$t_r = t - R(t_r)\,/c$

Calculation of the fields; assigned as a problem in WT;
details in Jackson; an involved calculation;

$$\mathbf{E}(\mathbf{x}, t) = e\left[\frac{(\mathbf{n} - \boldsymbol{\beta})(1 - \beta^2)}{\kappa^3 R^2}\right]_{\rm ret} +$$

$$+ \frac{e}{c}\left[\frac{\mathbf{n}}{\kappa^3 R} \times \{(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}\right]_{\rm ret}$$

and $\vec{B}(\vec{x},t) = \hat{n} \times \vec{E}(\vec{x},t)$
(everywhere)

14.2 Total Power Radiated by an Accelerated Charge—Larmor's Formula and Its Relativistic Generalization

If a charge is accelerated but is observed in a reference frame where its velocity is small compared to that of light, then in that coordinate frame the acceleration field in (14.14) reduces to

$$\mathbf{E}_a = \frac{e}{c}\left[\frac{\mathbf{n} \times (\mathbf{n} \times \dot{\boldsymbol{\beta}})}{R}\right]_{\text{ret}} \tag{14.18}$$

The instantaneous energy flux is given by the Poynting's vector,

$$\mathbf{S} = \frac{c}{4\pi}\,\mathbf{E} \times \mathbf{B} = \frac{c}{4\pi}\,|\mathbf{E}_a|^2\mathbf{n} \tag{14.19}$$

This means that the power radiated per unit solid angle is*

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}\,|R\mathbf{E}_a|^2 = \frac{e^2}{4\pi c}\,|\mathbf{n} \times (\mathbf{n} \times \dot{\boldsymbol{\beta}})|^2 \tag{14.20}$$

* In writing angular distributions of radiation we will always exhibit the polarization explicitly by writing the absolute square of a vector which is proportional to the electric field.

If Θ is the angle between the acceleration $\dot{\mathbf{v}}$ and $\mathbf{n}$, as shown in Fig. 14.3, then the power radiated can be written

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\,\dot{v}^2 \sin^2\Theta \tag{14.21}$$

This exhibits the characteristic $\sin^2\Theta$ angular dependence which is a well-known result. We note from (14.18) that the radiation is polarized in the plane containing $\dot{\mathbf{v}}$ and $\mathbf{n}$. The total instantaneous power radiated is obtained by integrating (14.21) over all solid angle. Thus

$$P = \frac{2}{3}\frac{e^2\dot{v}^2}{c^3} \tag{14.22}$$

This is the familiar Larmor result for a nonrelativistic, accelerated charge.

$$P = \frac{2}{3}\frac{e^2}{c}\gamma^6[(\dot{\boldsymbol{\beta}})^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2]$$

Fig. 14.4 Radiation pattern for charge accelerated in its direction of motion. The two patterns are not to scale, the relativistic one (appropriate for $\gamma \sim 2$) having been reduced by a factor $\sim 10^2$ for the same acceleration.