Exercise 12.3.3

Scattering by a conducting cylinder, with length 2L and radius a.

Exercise 12.3.3. Consider short wavelength electromagnetic scattering off of a perfectly conducting cylinder of radius a and length $2L$ oriented with its long axis perpendicular to the incoming plane wave. See Figure 12.7. ($\hat{n}'$ is the surface outward normal.) Choose your $z-$axis along the axis of the rod and your $x-$axis pointing in the direction of $\vec{k}_0$. Evaluate the illuminated side scattering amplitude for arbitrary $\vec{k}$ and show that the unpolarized cross section is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{ill}} = \frac{ka}{\pi}\sin(\phi/2)\left(\frac{\sin(kL\cos\theta)}{k\cos\theta}\right)^2,$$

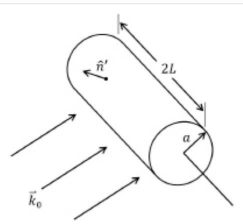


Fig. 12.7 Reference figure for Exercise 12.3.3.

where φ and θ are the usual spherical coordinate angles.

▪1▪ Draw a picture of the problem.

▪2▪ Write down eq 12.68.

Calculate the polarized cross section, starting from the surface integral over the illuminated side of the cylinder, equation (12.68):

$$A = \hat{\epsilon}_f \,.\, \vec{f}_{\text{ill}} = \frac{k}{4\pi i}\int_{\text{ill}} da' \exp\left\{-i(\vec{k}-\vec{k}_0).\,\vec{x}'\right.$$

$$\hat{\epsilon}_f \,.\, [\,(\hat{k} - \hat{k}_0)\times(\,\hat{n}' \times \hat{\epsilon}_0) - \hat{k}_0(\,\hat{n}' \cdot \hat{\epsilon}_0)\,]$$

▪3▪ Let $A = \hat{\epsilon}_f \bullet \vec{f}$. Then the polarized cross section is $\left(\frac{d\sigma}{d\Omega}\right)_{fi} = |\,A\,|^2$ and the unpolarized cross section is $\frac{1}{2}\sum_{j=1}^{2}\sum_{i=1}^{2}|\,A\,|^2$.

Geometry (see the Figure)

The z-axis is the axis of the cylinder.

The ends of the cylinder are parallel to the xy-plane.

The incident waves are moving parallel to the x-axis.

$\vec{k}_0 = k\,\hat{e}_x = k\{\,1, 0, 0\,\}$

$\epsilon_0^{[1]} = \hat{e}_y = \{0,1,0\}$ and $\epsilon_0^{[2]} = \hat{e}_z = \{0,0,1\}$.

$\vec{k} = k\{\,\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta\,\}$

$\epsilon_f^{[1]} = \{\,-\sin\phi,\,\cos\phi,\,0\,\}$

$\epsilon_f^{[2]} = \{\,-\cos\theta\cos\phi,\,-\cos\theta\sin\phi,\,\sin\theta\,\}$

▪4▪

Use cylindrical coordinates to do the surface integral.

Use cylindrical coordinates to do the surface integral.

$$\int_{\text{ill}} da' = \int_{-L}^{L} dz' \cdot \int_{\pi/2}^{3\pi/2} a\,d\phi'$$

Also, a point on the cylinder surface has ρ'=a, so

$\vec{x}' = \{\,a\cos\phi'\,,\,a\sin\phi',\,z'\,\}$

$$A = \hat{\epsilon}_f \,.\, \vec{f}_{\text{ill}} = \frac{k}{4\pi i}\,\mathrm{I}z \cdot \mathrm{I}\phi$$

where

$\text{Iz} = \int_{-L}^{L} dz' \exp\{-i\, k \cos\theta\, z'\}$

and

$\text{I}\phi = \int_{\pi/2}^{3\pi/2} a\, d\phi'$

∘ $\exp\{-i\, ka \sin\theta \cos(\phi-\phi') + i\, ka \cos\phi'\}$

∘ $\hat{\epsilon}_f \,.\, [\,(\hat{k} - \hat{e}_x) \times (\,\hat{n}' \times \hat{\epsilon}_0\,) - \hat{e}_x(\,\hat{n}' \cdot \hat{\epsilon}_0)\,]$

The polarized cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{f,i} = |\, \hat{\epsilon}_f \,.\, \vec{f}_{\text{ill}}\, |^2$$

$$= \frac{k^2}{(4\pi)^2} \circ |\, \text{Iz}\, |^2 \circ |\, \text{I}\phi\, |^2$$

▪2▪ The z′ integral

$$\text{Iz} = \frac{2 \sin(kL \cos\theta)}{k \cos\theta}$$

▪2▪ The ϕ′ integral

One approximation we can make in the ϕ' integral is to set $\theta = \pi/2$. That is because Iz is strongly peaked at $\cos\theta = 0$. So the first approximation is

$\text{I}\phi \approx \int_{\pi/2}^{3\pi/2} a\, d\phi'$

∘ $\exp\{-i\, ka\, [\,\cos(\phi-\phi') - \cos\phi'\,]\,\}$

∘ $\hat{\epsilon}_f \,.\, [\,(\hat{k} - \hat{e}_x) \times (\,\hat{n}' \times \hat{\epsilon}_0\,) - \hat{e}_x(\,\hat{n}' \cdot \hat{\epsilon}_0)\,]$

▪5▪ To do the ϕ' integral we need to use the stationary phase approximation. See pages 653 and 654. The stationary phase point ($\phi' = \phi'_{sp}$) is the point where the direction of $\vec{k}$ matches reflection by geometrical optics. Therefore, in the integrand we can set $\theta = \pi/2$; and we can set $\hat{n}' = \hat{n}'_r = \{\cos(\phi+\alpha), \sin(\phi+\alpha), 0\}$ where $\phi + 2\alpha = \pi$.

Now in the exponent, let $\phi' = \phi'_{sp} + \delta$; Taylor expand the exponent to order δ^2 ; and evaluate the Gaussian integral.

To do the ϕ' integral, we'll use the stationary phase approximation. See the text after equation (12.68) on pages 653 and 654. The calculation in the text is for a sphere.
The calculation in this problem is for a cylinder. But we've already separated off the z′ integral, so for this problem we only need to do a calculation for a circle.

As explained in the text, in the limit $ka \longrightarrow \infty$ the position of the stationary phase is $\phi'_{sp} = \frac{\pi}{2} + \frac{\phi}{2}$; this is the point on the surface where we have reflection according to geometrical optics:

$\vec{k_0} = k\,\{1,0,0\}$

$\vec{k} = k\,\{\cos\phi, \sin\phi, 0\}$ [[θ is set to $\pi/2$]]

$n' \approx \hat{n}'_r = \{\cos(\phi+\alpha),\ \sin(\phi+\alpha),\ 0\}$

where $2\alpha + \phi = \pi$; i.e., $\alpha = \frac{\pi}{2} - \frac{\phi}{2}$;

$\hat{n}'_r = \{\cos(\frac{\pi}{2} + \frac{\phi}{2}),\ \sin(\frac{\pi}{2} + \frac{\phi}{2}), 0\} = \{-\sin\frac{\phi}{2}, \cos\frac{\phi}{2}, 0\}$

▪5a▪ The polarization sum and average

Let $\text{PS} \equiv \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \{\ \hat{\epsilon}_f(j) \cdot$

$[(\hat{k} - \hat{e}_x) \times (\,\hat{n}'_r \times \hat{\epsilon}_0(i)\,) - \hat{e}_x(\,\hat{n}'_r \cdot \hat{\epsilon}_0(i)\,)\,]\,\}^2$

We might as well calculate this by Mathematica.

```
In[296]:= Remove["Global`*"]
ex = {1, 0, 0};
ϵ0[1] = {0, 1, 0};
ϵ0[2] = {0, 0, 1};
np = { -sϕh , cϕh , 0};
ku = { cϕ , sϕ , 0 };
ϵf[1] = { -sϕ, cϕ, 0 };
ϵf[2] = { 0, 0, 1 };

In[◦]:= Do[
 vec1[i] = Cross[ku - ex, Cross[np, ϵ0[i]]];
 vec2[i] = (-1) * Dot[np, ϵ0[i]] * ex,
 {i, 1, 2}]
Do[Do[q[j, i] = Dot[ϵf[j], vec1[i] + vec2[i]],
   {j, 1, 2}], {i, 1, 2}];
PS = (1 / 2) * Sum[Sum[ q[j, i] ^ 2 ,
    {j, 1, 2}], {i, 1, 2}]
PS = PS /. {cϕ → cϕh ^ 2 - sϕh ^ 2, sϕ → 2 * sϕh * cϕh};
PS = PS /. {cϕh → Sqrt[1 - sϕh ^ 2]};
PS = PS // FullSimplify
```

Out[◦]= $\frac{1}{2}\left((-c\phi h\, s\phi - s\phi h + c\phi\, s\phi h)^2 + (c\phi\,(-s\phi h + c\phi\, s\phi h) - s\phi\,(-c\phi h - s\phi\, s\phi h))^2\right)$

Out[◦]= $4\, s\phi h^2$

Result,

$PS = 4 \sin^2(\phi/2)$

▪5b▪ The integral over ϕ'

This is the integral that we need,

$$K = \int_{\pi/2}^{3\pi/2} d\phi' \exp\{-i\, ka\, [\cos(\phi-\phi') - \cos\phi'\,]\}\ .$$

We'll estimate it by the stationary phase approximation. Write ϕ'

$= \phi'_{sp} + \delta$ where δ is small. Recall from the above that $\phi'_{sp} = \frac{\pi}{2} + \frac{\phi}{2}$.

So

$$\begin{aligned}
&\cos(\phi - \phi') - \cos(\phi') \\
&= \cos(\phi - \phi'_{sp} - \delta) - \cos(\phi'_{sp} + \delta) \\
&= \cos\left(\frac{\phi}{2} - \delta - \frac{\pi}{2}\right) - \cos\left(\frac{\phi}{2} + \delta + \frac{\pi}{2}\right) \\
&= \sin\left(\frac{\phi}{2} - \delta\right) + \sin\left(\frac{\phi}{2} + \delta\right) \\
&= 2\sin\left(\frac{\phi}{2}\right)\cos(\delta)
\end{aligned}$$

$$\begin{aligned}
K &= \int_{-\phi/2}^{\pi-\phi/2} d\delta\ \exp\{-i\, ka\, 2\sin(\phi/2)\cos(\delta)\ \} \\
&\approx \int_{-\pi/2}^{\pi/2} d\delta \exp\{-i\, 2ka \sin(\phi/2)\cos(\delta)\} \\
&\approx \exp\{-i\lambda\} \int_{-\infty}^{\infty} d\delta \exp\{+i\lambda\, \delta^2/2)\}
\end{aligned}$$

where $\lambda = 2ka\sin(\phi/2)$. Do the Gaussian integral,

$K \approx \exp\{-i\lambda\}\,(\pi/(-i\lambda/2))^{1/2}$

Thus,

$|K|^2 = \frac{2\pi}{\lambda}$

```
(* Mathematica calculation *)
Assuming[μ > 0, Integrate[Exp[-μ*x^2],
  {x, -Infinity, Infinity}]]
K = Exp[-I*λ]*Sqrt[2π*I/λ] // FullSimplify
K2 = K*Conjugate[K] // ComplexExpand
K2 /. {λ → 13579}
```

Out[•]= $\frac{\sqrt{\pi}}{\sqrt{\mu}}$

Out[•]= $e^{-i\lambda}\sqrt{2\pi}\sqrt{\frac{i}{\lambda}}$

Out[•]= $2\pi\sqrt{\frac{1}{\lambda^2}}\,\mathrm{Cos}\left[\frac{1}{2}\mathrm{Arg}\left[\frac{i}{\lambda}\right]\right]^2 + 2\pi\sqrt{\frac{1}{\lambda^2}}\,\mathrm{Sin}\left[\frac{1}{2}\mathrm{Arg}\left[\frac{i}{\lambda}\right]\right]^2$

Out[•]= $\frac{2\pi}{13579}$

In[•]:= ?Conjugate

Conjugate[z] or z* gives the complex conjugate of the complex number z. >>

▪6▪ Square, sum, and average over polarizations. This step can be made easy by using Mathematica.

⟹ the unpolarized cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\sum_{i,j=1}^{2} |A_{f,i}|^2$$

where

$$A_{f,i} = \frac{k}{4\pi i} \cdot \mathrm{Iz} \cdot \mathrm{I}\phi$$

$$\mathrm{Iz} = \frac{2\sin(kL\cos\theta)}{k\cos\theta}$$

$$\frac{1}{2}\sum_{i,j=1}^{2} |\mathrm{I}\phi|^2 = a^2 \cdot |K|^2 \cdot \mathrm{PS}$$

$$= a^2 \cdot \frac{2\pi}{\lambda} \cdot 4\sin^2(\phi/2)$$

where $\lambda = 2ka\sin(\phi/2)$;

so $\frac{1}{2}\sum_{i,j=1}^{2} |\mathrm{I}\phi|^2 = \frac{8\pi a^2}{2ka\sin(\phi/2)}\sin^2(\phi/2)$

$$= \frac{4\pi a}{k}\sin(\phi/2)$$

Putting everything together,

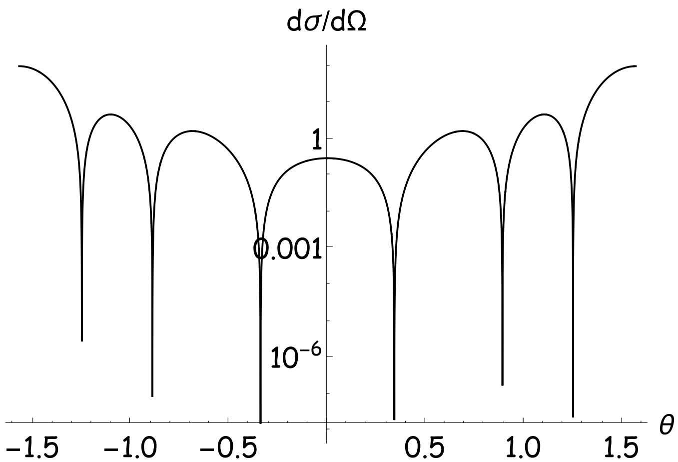
$$\frac{d\sigma}{d\Omega} = \frac{k^2}{16\pi^2}\, 4\left(\frac{\sin(kL\cos\theta)}{k\cos\theta}\right)^2 \frac{4\pi a}{k}\sin(\phi/2)$$

$$= \frac{ka}{\pi}\sin(\phi/2)\left(\frac{\sin(kL\cos\theta)}{k\cos\theta}\right)^2,$$

as claimed.

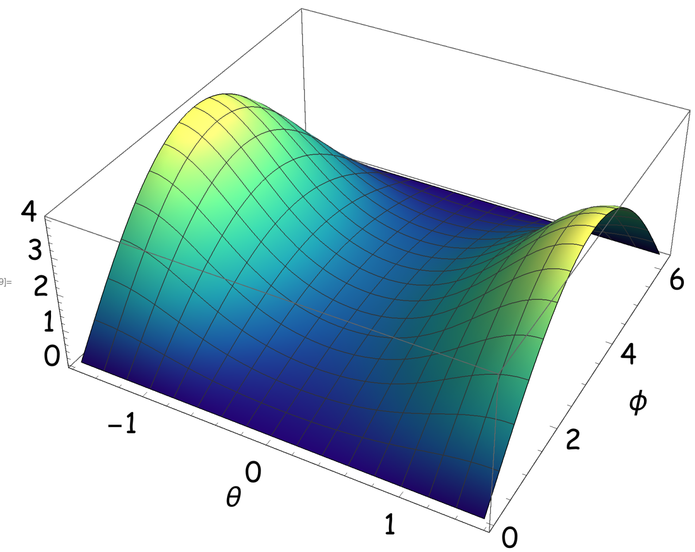
In[95]:=
```
LogPlot[Sin[10*Cos[th]]^2/Cos[th]^2, {th, -Pi/2, Pi/2},
  PlotRange → {All, All}, PlotPoints → 400,
  BaseStyle → "Text",
  PlotStyle → Black, AxesLabel → {"θ", "dσ/dΩ"},
  ImageSize → Large]
```

Out[95]=



In[109]:=
```
Plot3D[Sin[ph/2]*Sin[2*Cos[th]]^2/Cos[th]^2,
  {th, -Pi/2, Pi/2}, {ph, 0, 2 Pi},
  PlotRange → {All, All, All}, PlotPoints → 40,
  BaseStyle → "Text", ColorFunction → "BlueGreenYellow",
  PlotStyle → Black, AxesLabel → {"θ", "ϕ"},
  ImageSize → Large]
```

Out[109]=



In[99]:= ?Plot3D

Plot3D[f, {x, x_{min}, x_{max}}, {y, y_{min}, y_{max}}] generates a three–dimensional plot of f as a function of x and y.
Plot3D[{f_1, f_2, …}, {x, x_{min}, x_{max}}, {y, y_{min}, y_{max}}] plots several functions.
Plot3D[…, {x, y} ∈ reg] takes variables {x, y} to be in the geometric region reg. ≫

Problem 12-5 is due Friday 16 November

Hints

Problem 12-5 is Exercise 12.3.3;
scattering from a short conducting cylinder.
Hints:

▪1▪ Draw a picture of the problem.

▪2▪ Write down eq 12.68.

▪3▪ Let A = $\hat{\epsilon}_f \bullet \vec{f}$. Then the polarized cross section is $\left(\frac{d\sigma}{d\Omega}\right)_{fi} = |A|^2$ and the unpolarized cross section is $\frac{1}{2}\sum_{j=1}^{2}\sum_{i=1}^{2} |A|^2$.

▪4▪
Use cylindrical coordinates to do the surface integral.
$\int_{ill} da' = \int_{-L}^{L} dz' \,▫\, \int_{\pi/2}^{3\pi/2} a\, d\phi'$
Also, a point on the cylinder surface has ρ'=a, so
$\vec{x}' = \{ a\cos\phi' , a\sin\phi', z' \}$
$A = \hat{\epsilon}_f . \vec{f}_{ill} = \frac{k}{4\pi i}$ Iz ▫ Iϕ
where
$Iz = \int_{-L}^{L} dz' \exp\{-i k \cos\theta\, z' \}$
and
$I\phi = \int_{\pi/2}^{3\pi/2} a\, d\phi'$
▫ $\exp\{-i ka \sin\theta \cos(\phi-\phi') + i ka \cos\phi' \}$

▫ $\hat{\epsilon}_f . [(\hat{k} - \hat{e}_x) \times (\hat{n}' \times \hat{\epsilon}_0) - \hat{e}_x(\hat{n}' \bullet \hat{\epsilon}_0)]$

▪5▪ To do the ϕ' integral we need to use the stationary phase approximation. See pages 653 and 654. The stationary phase point ($\phi' = \phi'_{sp}$) is the point where the direction of $\vec{k}$ matches reflection by geometrical optics. Therefore, in the integrand we can set $\theta = \pi/2$; and we can set $\hat{n}' = \hat{n}'_r = \{ \cos(\phi+\alpha), \sin(\phi+\alpha), 0\}$ where $\phi + 2\alpha = \pi$.
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