Homework Assignment 2

Problem 2 (Exercise 1.13)

Given $\rho(\vec{x}, t) = e f(\vec{x} - \vec{R}(t))$

$$\vec{J}(\vec{x}, t) = e \frac{d\vec{R}}{dt} f(\vec{x} - \vec{R}(t))$$

where $f(\vec{\xi})$ is an arbitrary function.

Prove the continuity equation $(\nabla \cdot \vec{J} = -\partial\rho/\partial t)$

Proof $\quad \nabla \cdot \vec{J} = e \frac{d\vec{R}}{dt} \cdot \nabla f(\vec{x} - \vec{R}(t))$

and $\frac{\partial \rho}{\partial t} = e \nabla f(\vec{x} - \vec{R}(t)) \left(-\frac{d\vec{R}}{dt}\right)$ by the chain rule

$\therefore \quad \nabla \cdot \vec{J} = -\partial\rho/\partial t$. QED

Problem 3 (Exercise 5.4.2)

(a) Recall that a linear dielectric has

$\rho_{bound} = -\nabla\cdot\vec{P}$ and $\vec{D} = \vec{E} + 4\pi\vec{P} = \epsilon\vec{E}$.

Thus $\rho_{bound} = -\nabla\cdot\frac{1}{4\pi}(\vec{D}-\vec{E}) = -\nabla\cdot\frac{1}{4\pi}(\epsilon-1)\vec{E}$

$= \frac{1}{4\pi}(1-\epsilon)\,4\pi\,\rho_{total} = (1-\epsilon)(\rho_{bound}+\rho_{free})$

Thus $\epsilon\,\rho_{bound} = (1-\epsilon)\rho_{free}$

$\rho_{bound} = \left(\frac{1-\epsilon}{\epsilon}\right)\rho_{free}$

(b) The total bound charge is 0.

The bound charge on the surface is

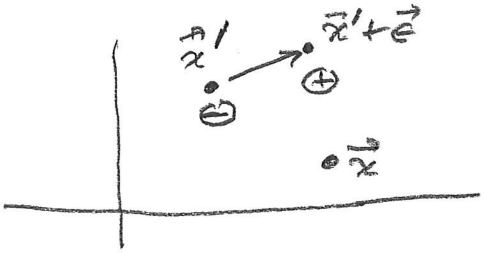
$$\oint_S \sigma_{bound}\,da = -\int_V \rho_{bound}\,d^3x = \left(\frac{\epsilon-1}{\epsilon}\right)\int_V \rho_{free}\,d^3x$$

$$\oint_S \sigma_{bound}\,da = \left(\frac{\epsilon-1}{\epsilon}\right)Q_{free}$$

Problem 4 (Exercise 5.4.4)

$(\vec{\epsilon} \to 0)$

(a)



$$\Phi(\vec{x}) = \frac{-q}{|\vec{x}-\vec{x}'|} + \frac{q}{|\vec{x}-\vec{x}'-\vec{\epsilon}|}$$

$$|\vec{x}-\vec{x}'-\vec{\epsilon}|^{-1} = \left\{ (\vec{x}-\vec{x}')^2 - 2\vec{\epsilon}\cdot(\vec{x}-\vec{x}') + \vec{\epsilon}^2 \right\}^{-1/2}$$

$$\approx |\vec{x}-\vec{x}'|^{-1} + \frac{\vec{\epsilon}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

So for a point like dipole

$$\Phi(\vec{x}) = \frac{q\vec{\epsilon}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \frac{\vec{p}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

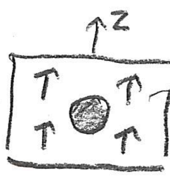
Now, for a dielectric material,

$$\Phi(\vec{x}) = \int \frac{d\vec{p}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \int d^3x' \, \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$= \int d^3x' \, \vec{P}(\vec{x}')\cdot\nabla' \frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \int_V d^3x' \, \frac{1}{|\vec{x}-\vec{x}'|}\left[-\nabla'\cdot\vec{P}(\vec{x}')\right] + \oint_S da' \, \frac{\hat{n}'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$ integration by parts

(b)



$\vec{P}(\vec{x}) = P_0\,\hat{e}_z$

Calculate the field inside the bubble.

$$\Phi(\vec{x}) = \int da' \, \frac{(-\hat{r}')\cdot\hat{e}_z P_0}{|\vec{x}-\vec{x}'|} = -P_0 a^2 \int \frac{\sin\theta' d\theta' d\phi' \cos\theta'}{|\vec{x}-\vec{x}'|}$$

One way to evaluate the integral is to use the addition theorem for $Y_{\ell m}$'s

$$\frac{1}{|\vec{x}-\vec{y}|} = \sum_{\ell m} \frac{r^\ell}{a^{\ell+1}} \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta\phi) Y_{\ell m}^*(\theta',\phi') \quad (4.225)$$

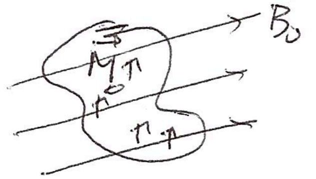
$$\text{Now } \int \sin\theta' d\theta' d\phi' \underbrace{\cos\theta'}_{\sqrt{\frac{4\pi}{3}} Y_{10}} Y_{\ell m}^*(\theta',\phi') = \sqrt{\frac{4\pi}{3}} \delta_{\ell 0} \delta_{m0}$$

so

$$\Phi(\vec{x}) = -P_0 a^2 \frac{r}{a^2} \frac{4\pi}{3} \sqrt{\frac{3}{4\pi}} \cos\theta \sqrt{\frac{4\pi}{3}} = -\frac{4\pi}{3} P_0 z$$

$$\vec{E}(\vec{x}) = -\nabla\Phi = \frac{4\pi}{3} P_0 \hat{e}_z .$$

Problem 5 (Exercise 6.10.1)



$\vec M(\vec x)$ is constant inside the material, $\vec M_0$

$\vec K_{eff} = c\vec M \times \hat n$ (surface current density)

$$\vec N = \oint_S da\ \vec x \times \left[\frac{1}{c}\vec K_{eff} \times \vec B_0\right]$$

(torque on a magnet)

(a)

To prove: $\vec N = \vec m \times \vec B_0$ where $\vec m = \int_V \vec M(\vec x)\, d^3x = V\vec M_0$

Proof

$$\vec N = \oint_S da\ \vec x \times [(\vec M \times \hat n) \times \vec B_0]$$

$$= \oint_S da \left\{ (\vec M \times \hat n)(\vec x \cdot \vec B_0) - [\underbrace{\vec x \cdot (\vec M \times \hat n)}]\vec B_0 \right\}$$

$=0$ because $\hat n$ and $\vec x$ are parallel

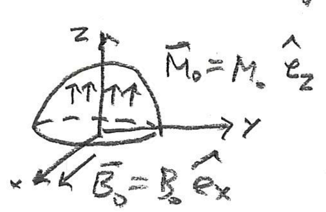
$$= \hat e_i\, \epsilon_{ijk}\, M_{0j} \Big(\underbrace{\oint_S da\, n_k x_l}\Big) B_{0l} \quad \text{(index notation)}$$

$= \int_V d^3x\, \nabla_k x_l$ by Gauss's law

$= V\delta_{kl}$

$$= \hat e_i\, \epsilon_{ijk} V M_{0j} B_{0l} = \vec m \times \vec B_0 \quad \text{where } \vec m = V\vec M_0$$

(b)



$$\left.\begin{aligned} \vec m &= \tfrac{2}{3}\pi a^3 M_0\, \hat e_z \\ \vec B_0 &= B_0\, \hat e_x \end{aligned}\right\} \quad \vec N = \tfrac{2}{3}\pi a^3 M_0 B_0\, \hat e_y$$

Problem 6

(a) $u_E = \frac{1}{8\pi} E^2$ and $u_M = \frac{1}{8\pi} B^2$

$$\frac{\partial}{\partial t}(u_E + u_M) = \frac{1}{4\pi}\Big(\vec{E}\cdot \underbrace{\frac{\partial \vec{E}}{\partial t}}_{\to\, c\nabla\times\vec{B} - 4\pi\vec{J}} + \vec{B}\cdot \underbrace{\frac{\partial \vec{B}}{\partial t}}_{\to\, -c\nabla\times\vec{E}}\Big)$$

microscopic field equations

$$\frac{\partial u}{\partial t} = \frac{c}{4\pi}\Big[\underbrace{\vec{E}\cdot\nabla\times\vec{B} - \vec{B}\cdot\nabla\times\vec{E}}_{-\nabla\cdot(\vec{E}\times\vec{B})}\Big] - \vec{J}\cdot\vec{E}$$

$$= -\nabla\cdot\vec{S} - \vec{J}\cdot\vec{E} \quad \text{where} \quad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$$

Conservation of energy :

$$\frac{\partial u}{\partial t} + \nabla\cdot\vec{S} + \vec{J}\cdot\vec{E} = 0$$

local work done by the fields + increase of field energy

(b) $$u = \frac{1}{8\pi}\vec{E}\cdot\vec{D} + \frac{1}{8\pi}\vec{B}\cdot\vec{H} = \frac{\epsilon}{8\pi}E^2 + \frac{1}{8\pi\mu}B^2$$

$$\frac{\partial u}{\partial t} = \frac{1}{4\pi}\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \frac{1}{4\pi}\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}$$

$$= \frac{1}{4\pi}\vec{E}\cdot\left\{c\,\nabla\times\vec{H} - 4\pi\vec{J}_{free}\right\} + \frac{1}{4\pi}\vec{H}\cdot(-c\,\nabla\times\vec{E})$$

$$= \frac{-c}{4\pi}\nabla\cdot(\vec{E}\times\vec{H}) - \vec{E}\cdot\vec{J}_{free}$$

Conservation of energy

$$\frac{\partial u}{\partial t} + \nabla\cdot\vec{S} + \vec{E}\cdot\vec{J}_{free} = 0 \quad \text{where} \quad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$$